Interval Constraints Overview

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DI/FCT/UNL

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Lecture 1: Interval Constraints Overview

Continuous Constraint Satisfaction Problems

Continuous Constraint Reasoning

- Representation of Continuous Domains
- Pruning and Branching

Solving Continuous CSPs

- Constraint Propagation
- Consistency Criteria

Applications

Course Structure

Constraint Reasoning

Continuous CSP (CCSP):

Constraint Satisfaction Problem (CSP):

set of variables

set of domains → Intervals of reals [a,b]

set of constraints → Numeric $(=,\leq,\geq)$

Solution: → Many

assignment of values which satisfies all the constraints

GOAL Find Solutions;
Find an enclosure of the solution space

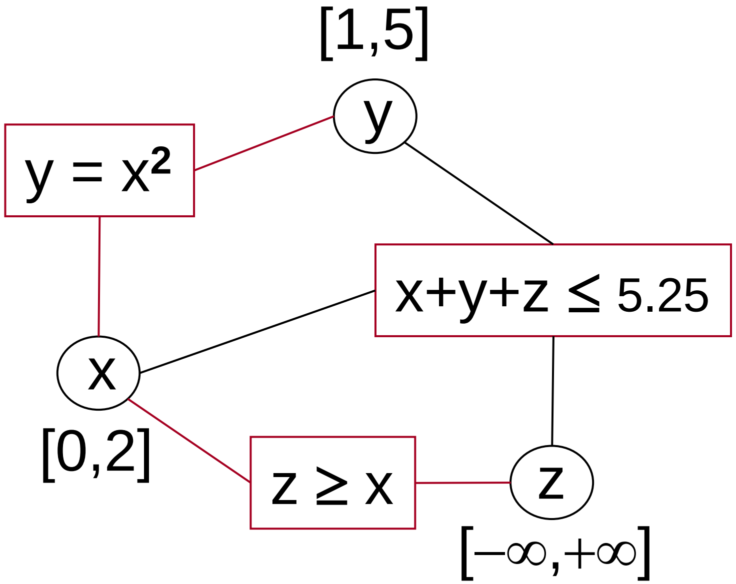
Constraint Reasoning

Continuous Constraint Satisfaction Problem (CCSP):

Interval Domains

Numerical Constraints

Many Solutions



Solution:

x=1, y=1, z=1

...

x=1, y=1, z=3.25

...

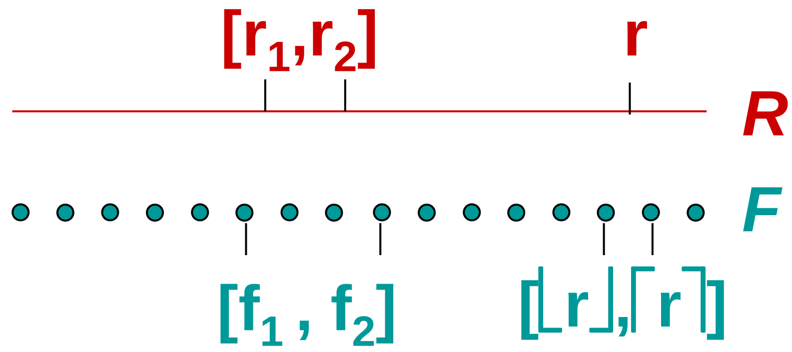
assignment of values which satisfies all the constraints

GOAL Find solutions;

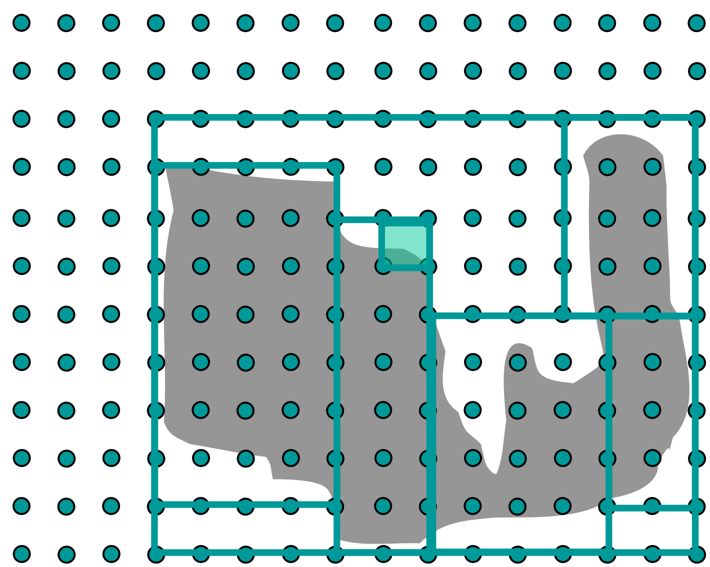
Find an enclosure of the solution space

Representation of Continuous Domains

***F*-interval**



***F*-box**



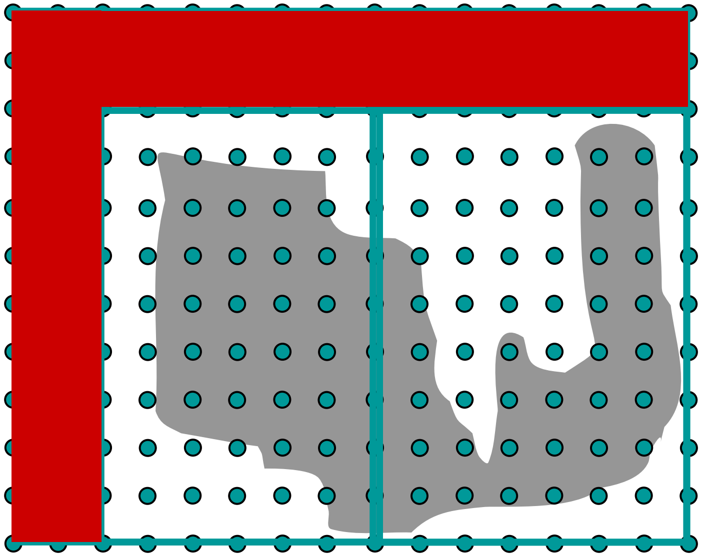
Canonical solution

Solving CCSPs:

Branch and Prune algorithms

constraint propagation

box split



Safe Narrowing Functions

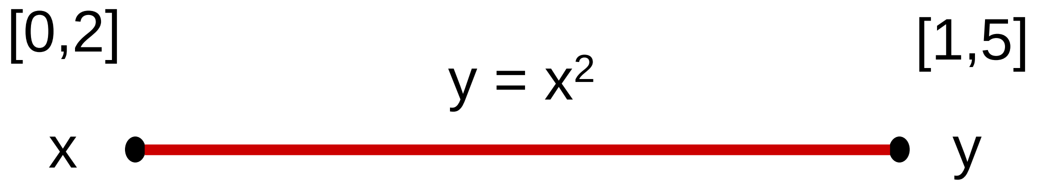
Strategy for
- isolate canonical solutions
- provide an enclosure of the solution space

depends on a consistency requirement

Constraint Reasoning (vs Simulation)

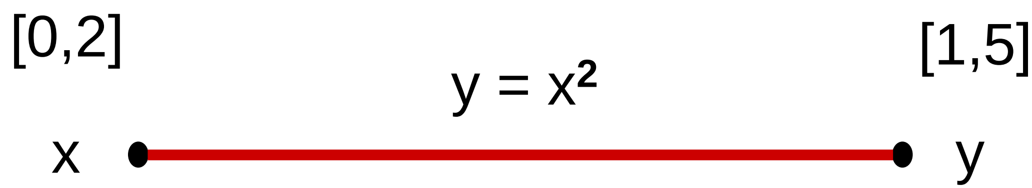
Represents uncertainty as intervals of possible values

Uses safe methods for narrowing the intervals accordingly to the constraints of the model



		x	y	
Simulation:		0	0	no
	x≤1?	1	1	
		2	4	y≥4?
Constraint Reasoning:		[1,2]	[1,4]	

How to narrow the domains?



Safe methods are based on Interval Analysis techniques

$$\forall_{x\in[a,b]}\ x^2 \in [a,b]^2 = \begin{cases} [0, \lceil \max(a^2,b^2) \rceil] & \text{if } a \le 0 \le b \\ [\lfloor \min(a^2,b^2) \rfloor, \lceil \max(a^2,b^2) \rceil] & \text{otherwise} \end{cases}$$

If $x \in [0,2]$

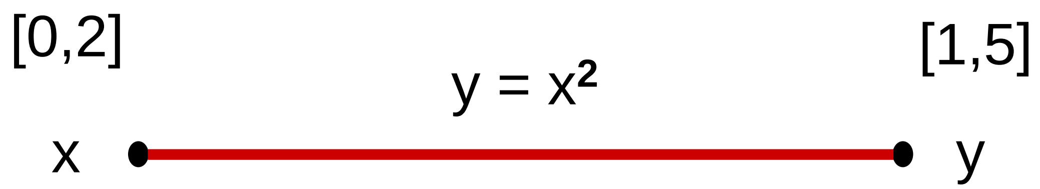
Then $y \in [0,2]^2 = [0, \lceil \max(0^2,2^2) \rceil] = [0,4]$

$\therefore\ y \in [1,5] \wedge y \in [0,4]$

$\therefore\ y \in [1,5] \cap [0,4]$

$\therefore\ y \in [1,4]$

How to narrow the domains?



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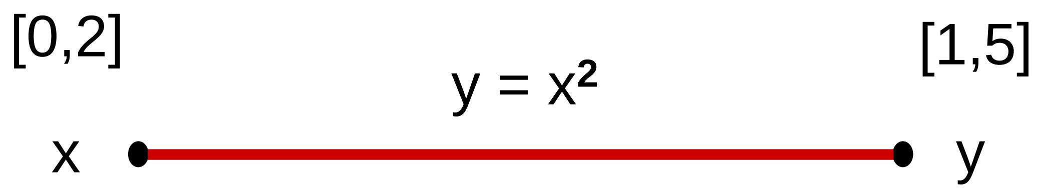
$$\forall_{x\in[a,b]}\ x^2 \in [a,b]^2 = \begin{cases} [0, \lceil \max(a^2,b^2) \rceil] & \text{if } a \le 0 \le b \\ [\lfloor \min(a^2,b^2) \rfloor, \lceil \max(a^2,b^2) \rceil] & \text{otherwise} \end{cases}$$

$$NF_{y=x^2}: Y' \leftarrow Y \cap X^2$$

How to narrow the domains?



Safe methods are based on Interval Analysis techniques

$y - x^2 = 0 \longrightarrow F(Y) = Y - [0,2]^2 \qquad F'(Y) = 1$

$$\forall_{y \in Y} \forall_{x \in [0,2]}\ y - x^2 = 0 \Rightarrow y \in N(Y) = c(Y) - \frac{F(c(Y))}{F'(Y)}$$

Interval Newton method

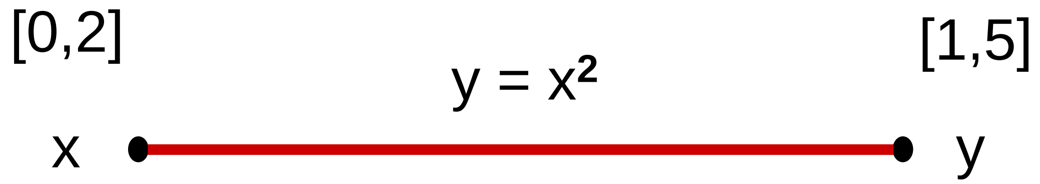
If $x \in [0,2]$ and $y \in [1,5]$

Then $y \in N([1,5]) = 3 - \dfrac{3 - [0,2]^2}{1} = [0,4]$

$\therefore\ y \in [1,5] \cap [0,4]$

$\therefore\ y \in [1,4]$

How to narrow the domains?



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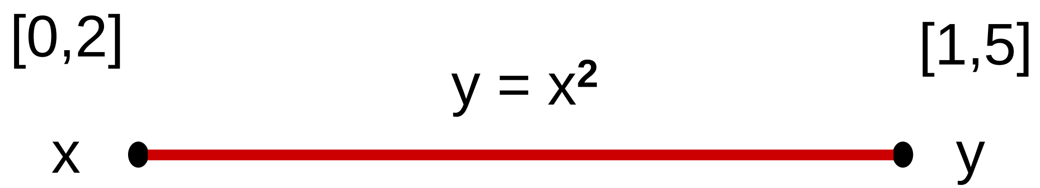
Interval Newton method

$$NF_{y=x^2}:\ Y' \leftarrow Y \cap \left(c(Y) - \frac{c(Y) - X^2}{1} \right)$$

How to narrow the domains?

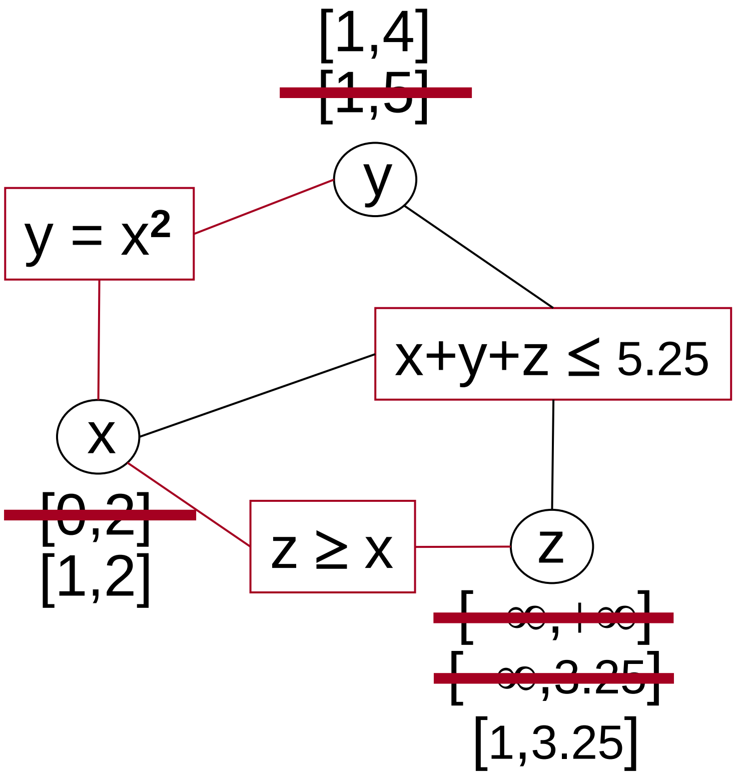


Safe methods are based on Interval Analysis techniques

	contractility	correctness
$NF_{y=x^2}: Y' \leftarrow Y \cap X^2$	$Y' \subseteq Y$	$\forall_{y \in Y}\, y \notin Y' \Rightarrow \neg\exists_{x \in X}\, y=x^2$
$NF_{y=x^2}: Y' \leftarrow Y \cap \left(c(Y) - \dfrac{c(Y) - X^2}{1} \right)$		
$NF_{y=x^2}: X' \leftarrow (X \cap -Y^{1/2}) \overset{+}{\cup} (X \cap +Y^{1/2})$	$X' \subseteq X$	$\forall_{x \in X}\, x \notin X' \Rightarrow \neg\exists_{y \in Y}\, y=x^2$
$NF_{y=x^2}: X' \leftarrow X \cap \left(c(X) - \dfrac{Y - c(X)^2}{-2X} \right)$		

Solving a Continuous Constraint Satisfaction Problem

Constraint Propagation



$NF_{y=x^2}: Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}: X' \leftarrow (X \cap -Y^{1/2}) \uplus (X \cap +Y^{1/2})$

$NF_{x+y+z \leq 5.25}: X' \leftarrow X \cap ([-\infty, 5.25] - Y - Z)$

$NF_{x+y+z \leq 5.25}: Y' \leftarrow Y \cap ([-\infty, 5.25] - X - Z)$

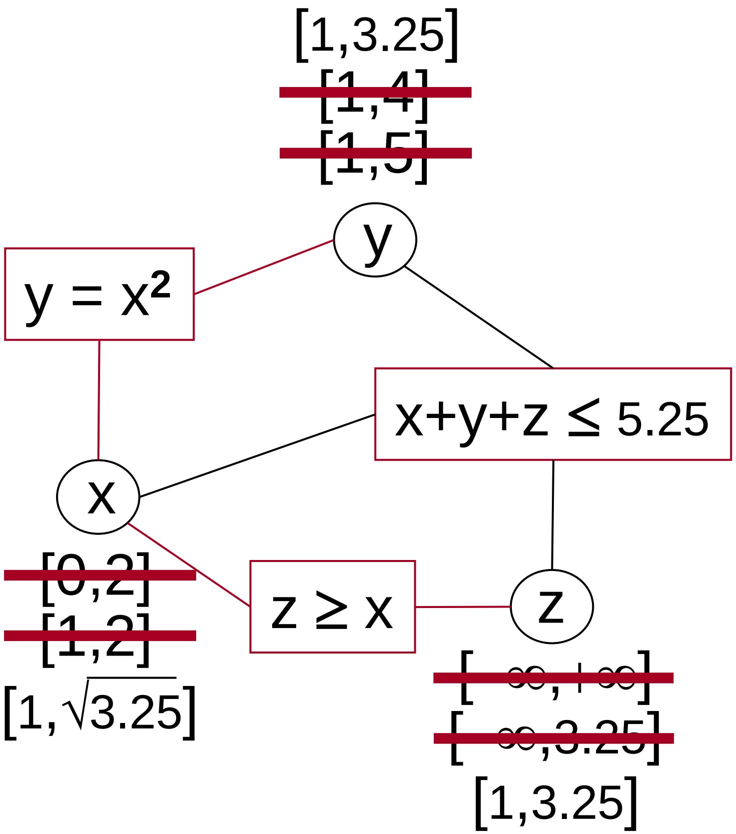
$NF_{x+y+z \leq 5.25}: Z' \leftarrow Z \cap ([-\infty, 5.25] - X - Y)$

$NF_{z \geq x}: X' \leftarrow X \cap (Z - [0, +\infty])$

$NF_{z \geq x}: Z' \leftarrow Z \cap (X + [0, +\infty])$

Solving a Continuous Constraint Satisfaction Problem

Constraint Propagation



√ $NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

√ $NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{1/2}) \uplus (X \cap +Y^{1/2})$

√ $NF_{x+y+z \leq 5.25}$: $X' \leftarrow X \cap ([-\infty, 5.25] - Y - Z)$

√ $NF_{x+y+z \leq 5.25}$: $Y' \leftarrow Y \cap ([-\infty, 5.25] - X - Z)$

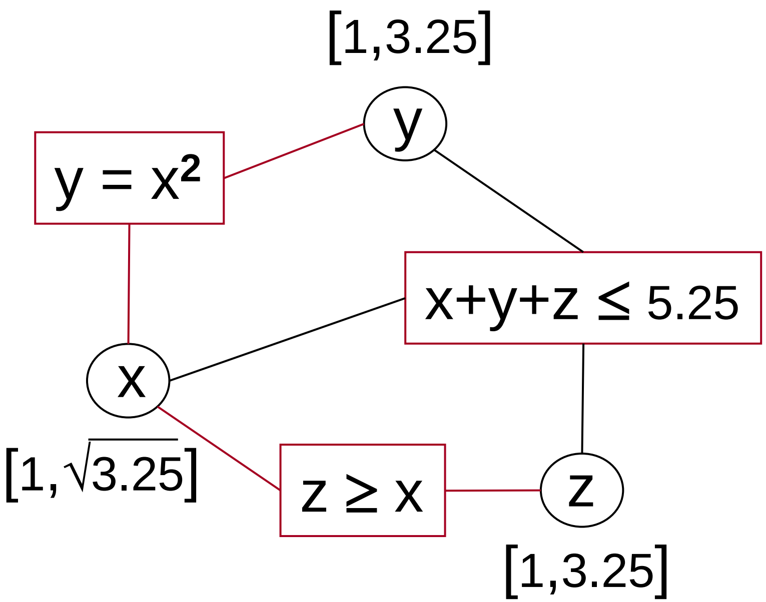
√ $NF_{x+y+z \leq 5.25}$: $Z' \leftarrow Z \cap ([-\infty, 5.25] - X - Y)$

√ $NF_{z \geq x}$: $X' \leftarrow X \cap (Z - [0, +\infty])$

√ $NF_{z \geq x}$: $Z' \leftarrow Z \cap (X + [0, +\infty])$

Solving a Continuous Constraint Satisfaction Problem

- Constraint Propagation + Branching
- Consistency Criterion



x	y	z	
1	1	1	√
1	1	3.25	√
$\sqrt{3.25}$ ✗			
1.5	2.25	1.5	√

$y = x^2 \Rightarrow y = 3.25$

$x+y+z \leq 5.25 \Rightarrow z \leq 2-\sqrt{3.25}$, where $2-\sqrt{3.25} < \sqrt{3.25}$

~~z ≥ x~~

Solving a Continuous Constraint Satisfaction Problem

{ Constraint Propagation + Branching
  Consistency Criterion

Local Consistency (2B-Consistency) ← Constraint Propagation

↓

Higher Order Consistencies (kB-Consistency) ← Constraint Propagation + Branching

3B-Consistency: if 1 bound is fixed then the problem is Local Consistent

x	y	z			x	y	z
$[1,\sqrt{3.25}]$	[1,3.25]	[1,3.25]	not 3B-Consistent	←	$[\sqrt{3.25}]$	[1,3.25]	[1,3.25]
[1,1.5]	[1,2.25]	[1,3.25]	3B-Consistent		not Local Consistent		

Example:

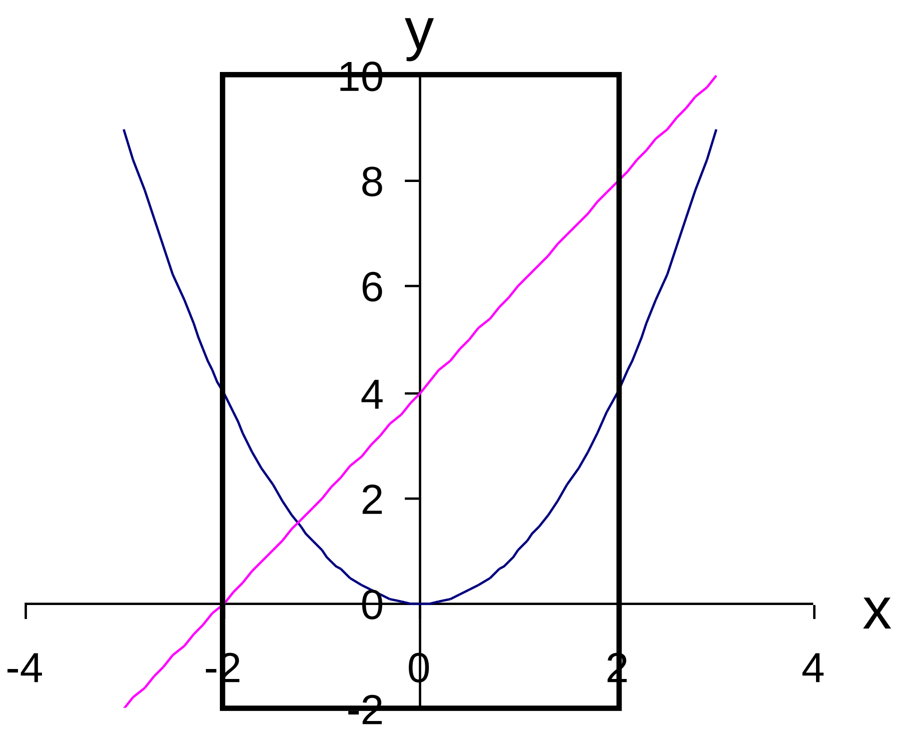
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

define set of narrowing functions:

$y = x^2$ → $NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$x = \pm y^{1/2}$ → $NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{1/2}) \uplus (X \cap +Y^{1/2})$

$y = 2x + [4,+\infty]$ → $NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$x = \frac{1}{2}y - [2,+\infty]$ → $NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (\frac{1}{2}Y - [2,+\infty])$

Example:

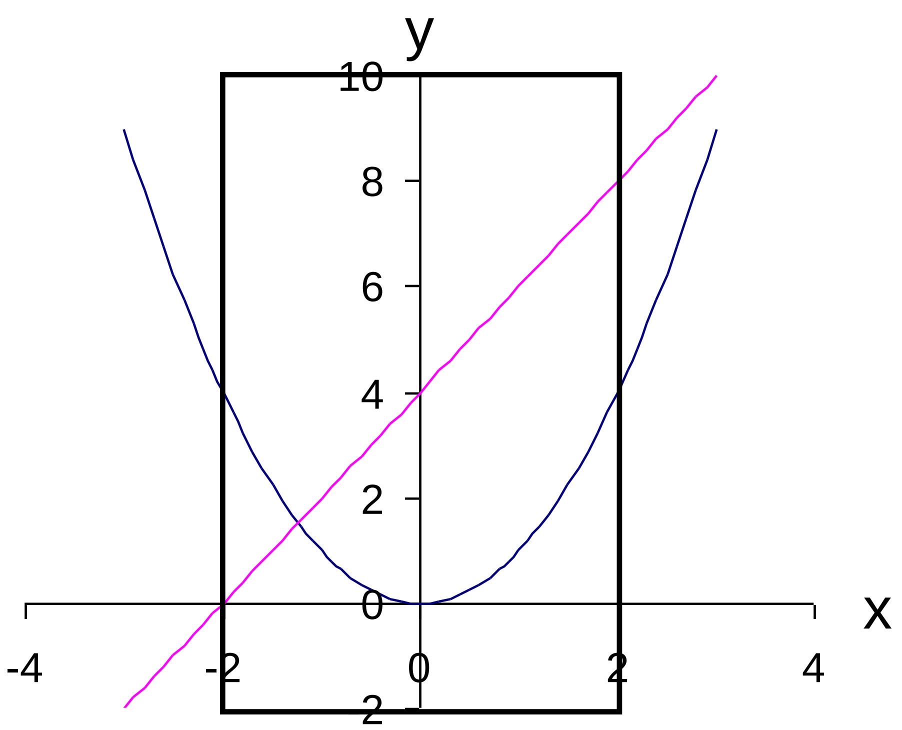
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: $[-2,2]\times[-2,10]$

$[-2,2]\times([-2,10]\cap[-2,2]^2)$ ← $NF_{y=x^2}: Y' \leftarrow Y \cap X^2$

$[-2,2]\times([-2,10]\cap[0,4])$ — $NF_{y=x^2}: X' \leftarrow (X \cap -Y^{½}) \uplus (X \cap +Y^{½})$

$[-2,2]\times[0,4]$ — $NF_{y\geq 2x+4}: Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}: X' \leftarrow X \cap (½Y-[2,+\infty])$

Example:

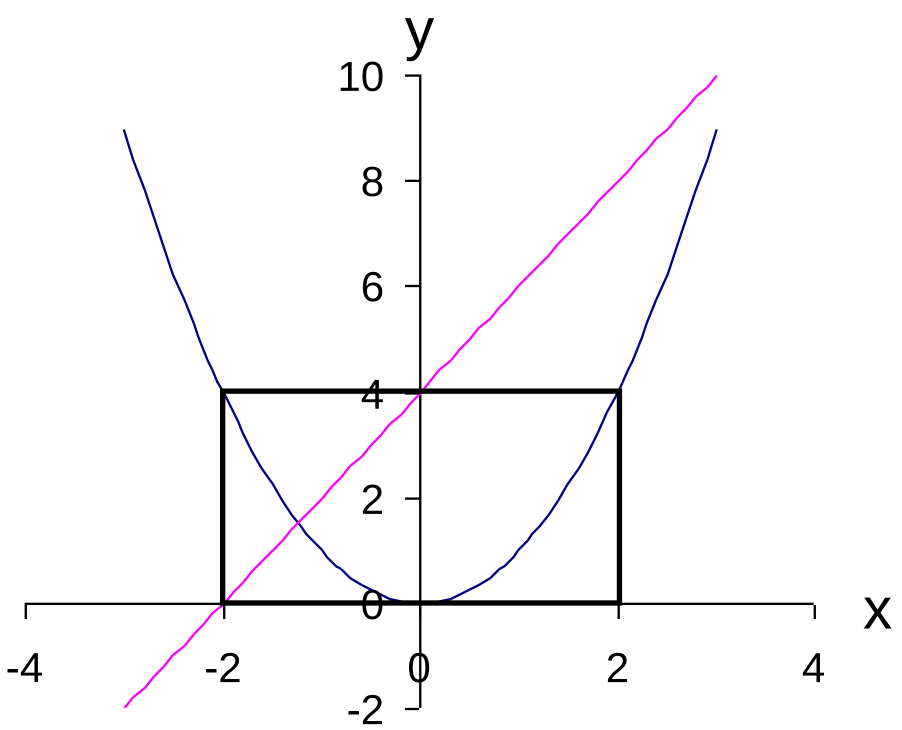
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: $[-2,2]\times[0,4]$

$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{1/2}) \cup (X \cap +Y^{1/2})$

$NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (\frac{1}{2}Y - [2,+\infty])$

$([-2,2]\cap -[0,4]^{1/2}) \uplus ([-2,2]\cap [0,4]^{1/2}) \times [0,4]$

$([-2,2]\cap[-2,0]) \uplus ([-2,2]\cap[0,2]) \times [0,4]$ ←

$[-2,2]\times[0,4]$

Example:

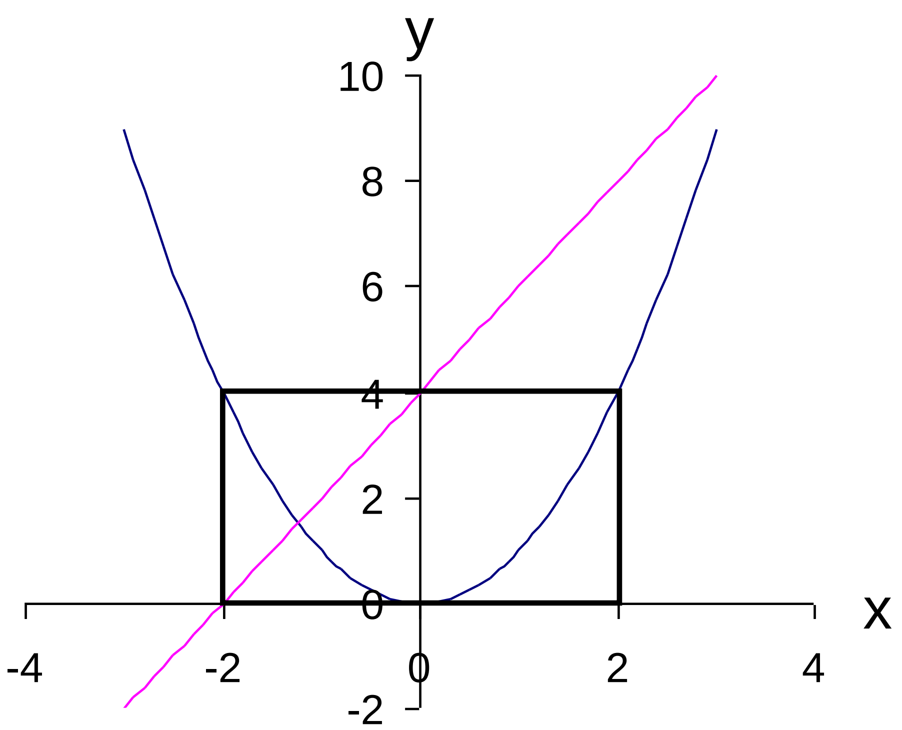
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: $[-2,2]\times[0,4]$

$[-2,2]\times([0,4]\cap(2[-2,2]+[4,+\infty]))$

$[-2,2]\times([0,4]\cap([-4,4]+[4,+\infty]))$

$[-2,2]\times([0,4]\cap[0,+\infty])$

$[-2,2]\times[0,4]$

$NF_{y=x^2}$: $Y' \leftarrow Y\cap X^2$

$NF_{y=x^2}$: $X' \leftarrow (X\cap -Y^{1/2})\cup(X\cap +Y^{1/2})$

$NF_{y\geq 2x+4}$: $Y' \leftarrow Y\cap(2X+[4,+\infty])$

$NF_{y\geq 2x+4}$: $X' \leftarrow X\cap(\frac{1}{2}Y-[2,+\infty])$

Example:

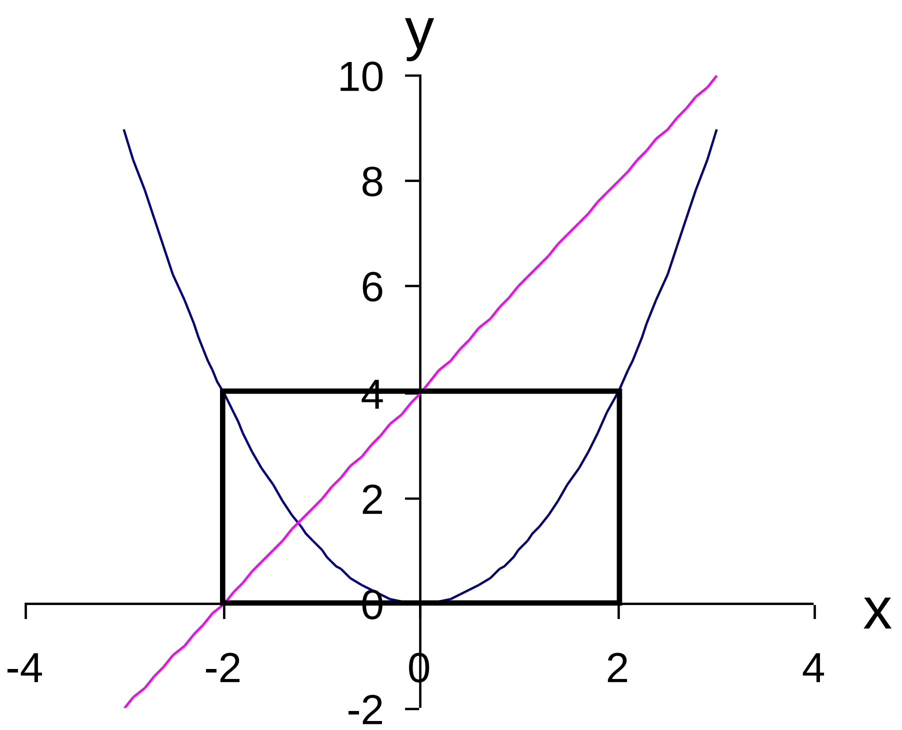
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: $[-2,2]\times[0,4]$

$([-2,2]\cap(\frac{1}{2}[0,4]-[2,+\infty]))\times[0,4]$ — $NF_{y=x^2}$: $Y' \leftarrow Y\cap X^2$

$([-2,2]\cap([0,2]-[2,+\infty]))\times[0,4]$ — $NF_{y=x^2}$: $X' \leftarrow (X\cap -Y^{1/2})\cup(X\cap +Y^{1/2})$

$([-2,2]\cap[-\infty,0])\times[0,4]$ — $NF_{y\geq 2x+4}$: $Y' \leftarrow Y\cap(2X+[4,+\infty])$

$[-2,0]\times[0,4]$ ← $NF_{y\geq 2x+4}$: $X' \leftarrow X\cap(\frac{1}{2}Y-[2,+\infty])$

Example:

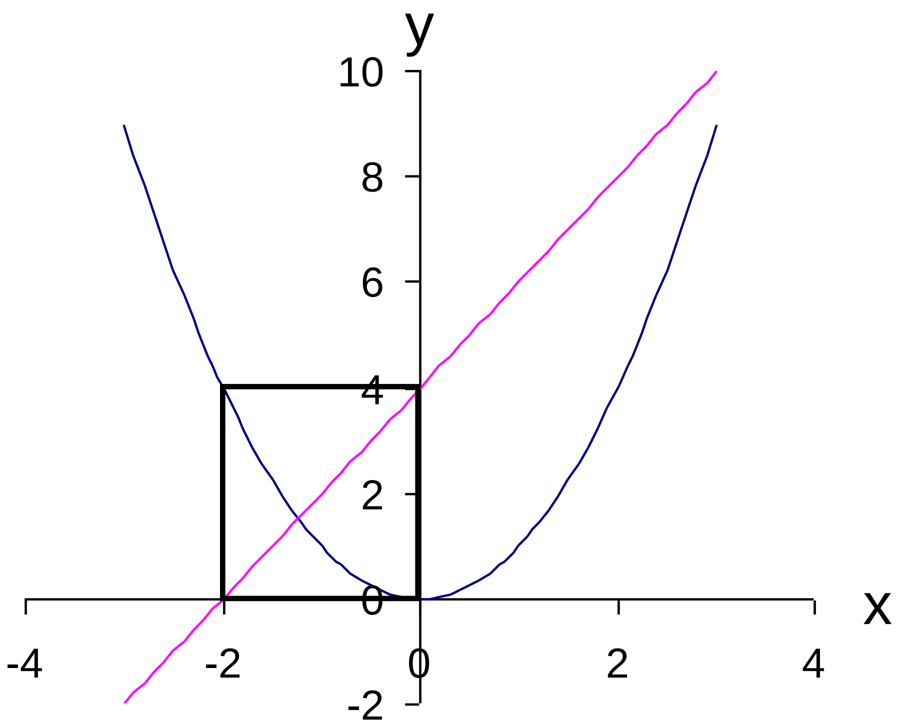
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

obtained the box: $[-2,0]\times[0,4]$ **(fixed point)**

$NF_{y=x^2}: Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}: X' \leftarrow (X \cap -Y^{1/2}) \cup (X \cap +Y^{1/2})$

$NF_{y\geq 2x+4}: Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}: X' \leftarrow X \cap (\frac{1}{2}Y-[2,+\infty])$

Example:

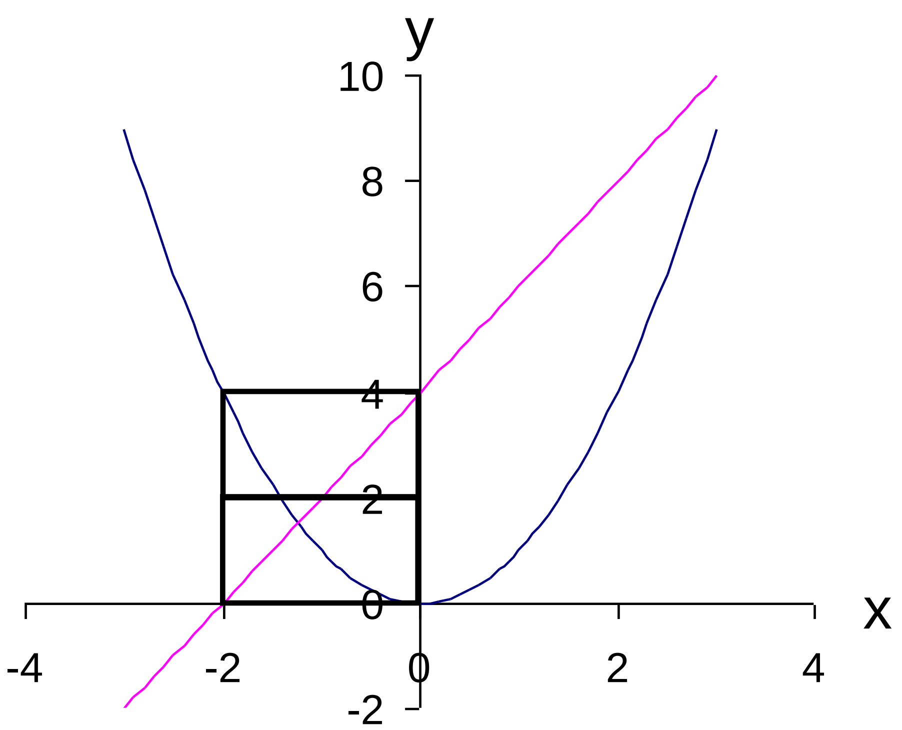
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Split box

[-2,0]×[0,2]

[-2,0]×[2,4]

Example:

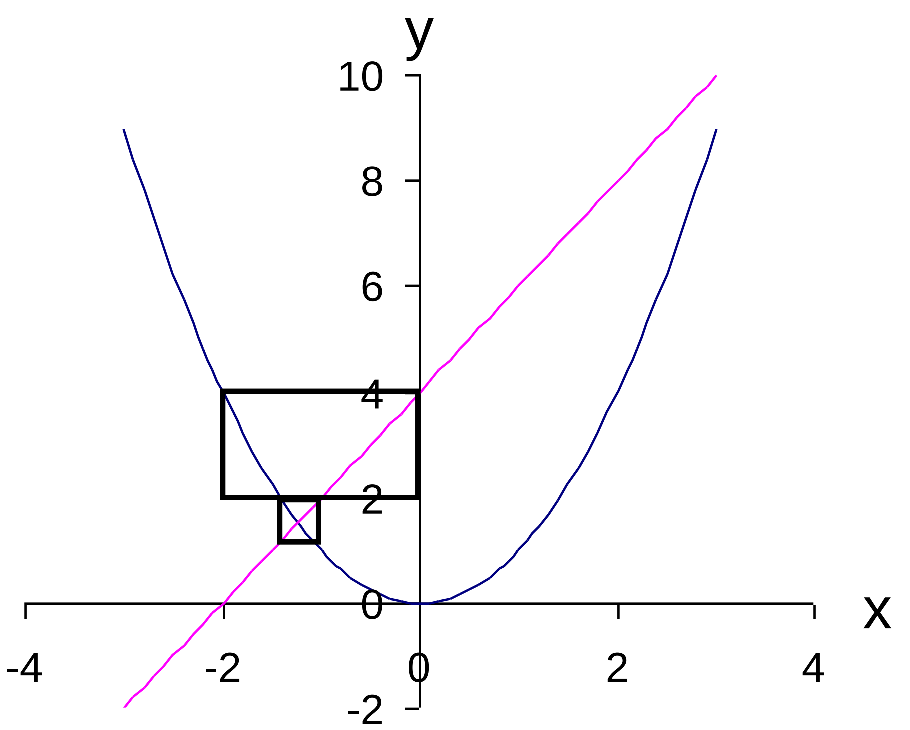
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Split box

[-2,0]×[0,2] —prune→ [-1.415,-1.082]×[1.171,2.000] (fixed point)

[-2,0]×[2,4]

Example:

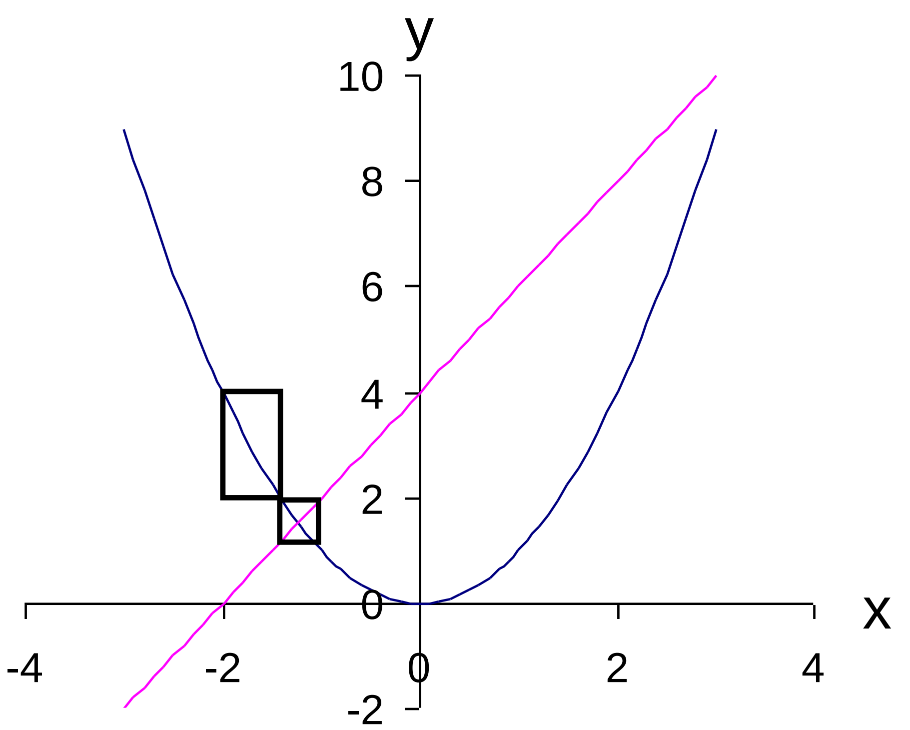
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Split box

[-2,0]×[0,2] —prune→ [-1.415,-1.082]×[1.171,2.000] (fixed point)

[-2,0]×[2,4] —prune→ [-2.000,-1.414]×[2.000,4.000] (fixed point)

Example:

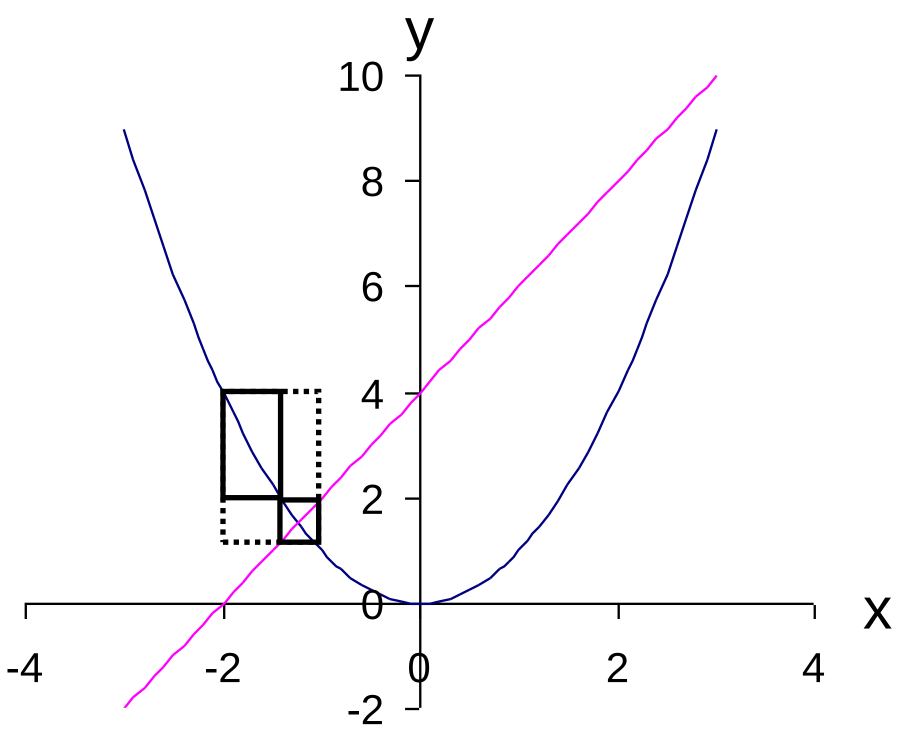
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



When to stop? → Stopping Criteria

if we stop now:

[-1.415,-1.082]×[1.171,2] ⊎ [-2,-1.414]×[2,4] = [-2 ,-1.082]×[1.171,4]

Example:

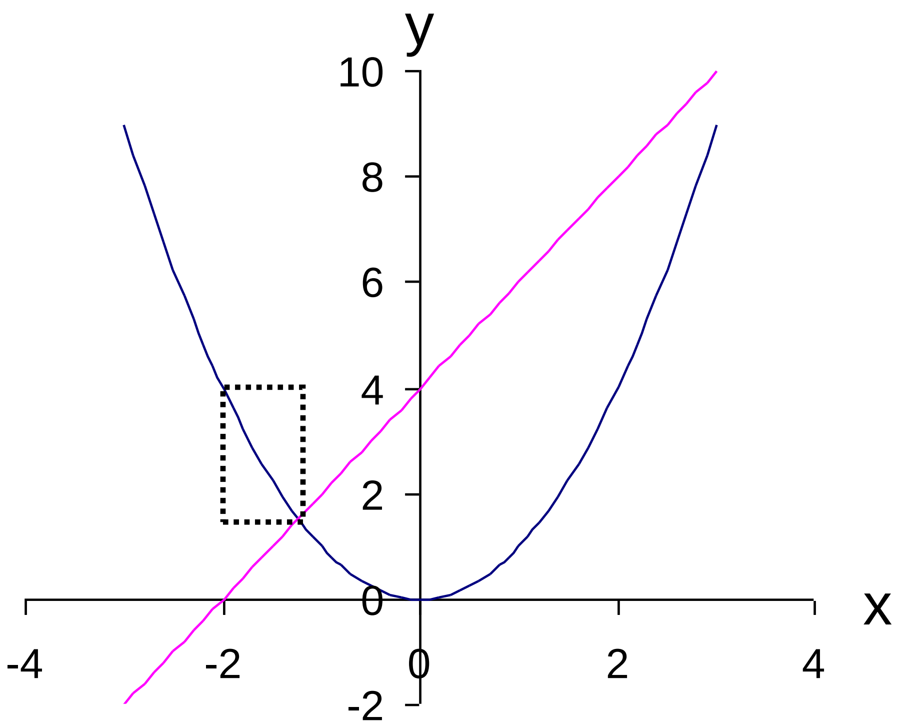
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



When to stop? → Stopping Criteria

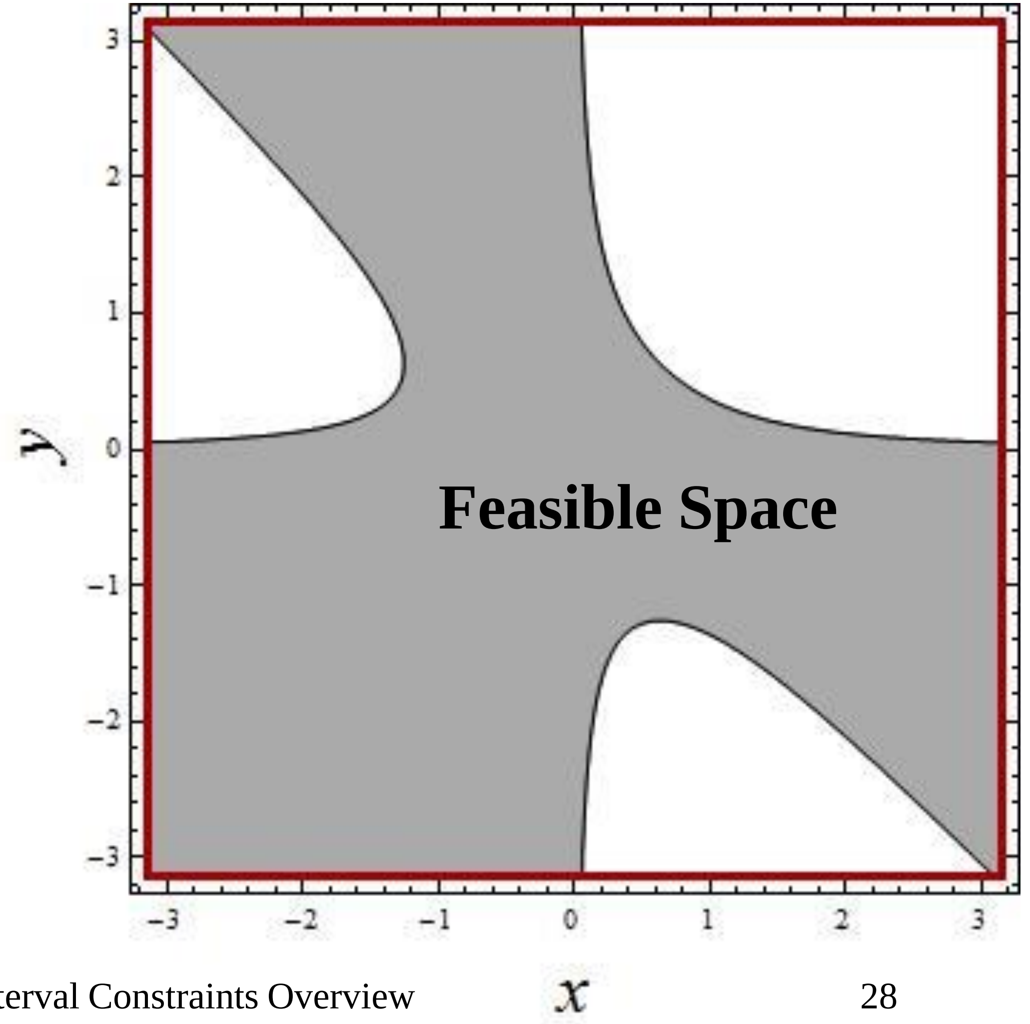
smallest box containing all canonical solutions

Continuous Constraint Programming

Continuous Constraint Satisfaction Problem:

$$x \in [-\pi, \pi] \quad y \in [-\pi, \pi]$$

$$x^2 y + xy^2 \leq 0.5$$

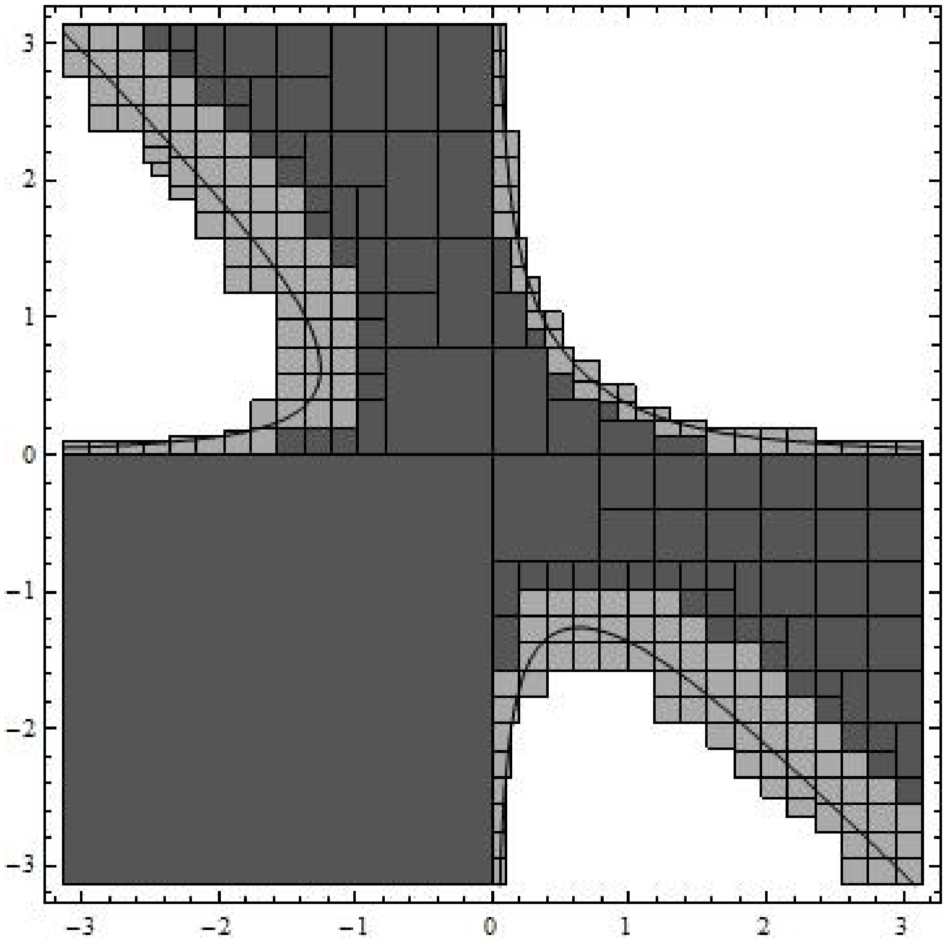


Continuous Constraint Programming

Continuous Constraint Satisfaction Problem:

$$x \in [-\pi, \pi] \quad y \in [-\pi, \pi]$$

$$x^2 y + xy^2 \leq 0.5$$



Branch & Prune Algorithms:

- **one** solution
- feasible space **box cover**

Prune Techniques:

- Interval Analysis
- Constraint Propagation

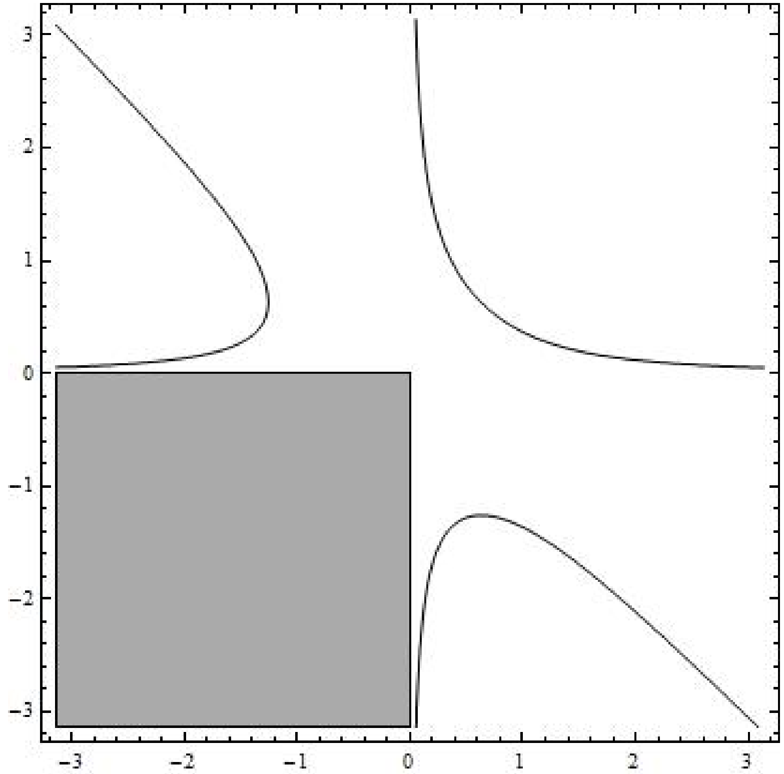
Interval Analysis

$$x^2 y + xy^2 \leq 0.5$$

Inner Box ?

$$x \in [-\pi, 0]$$

$$y \in [-\pi, 0]$$



Interval Analysis

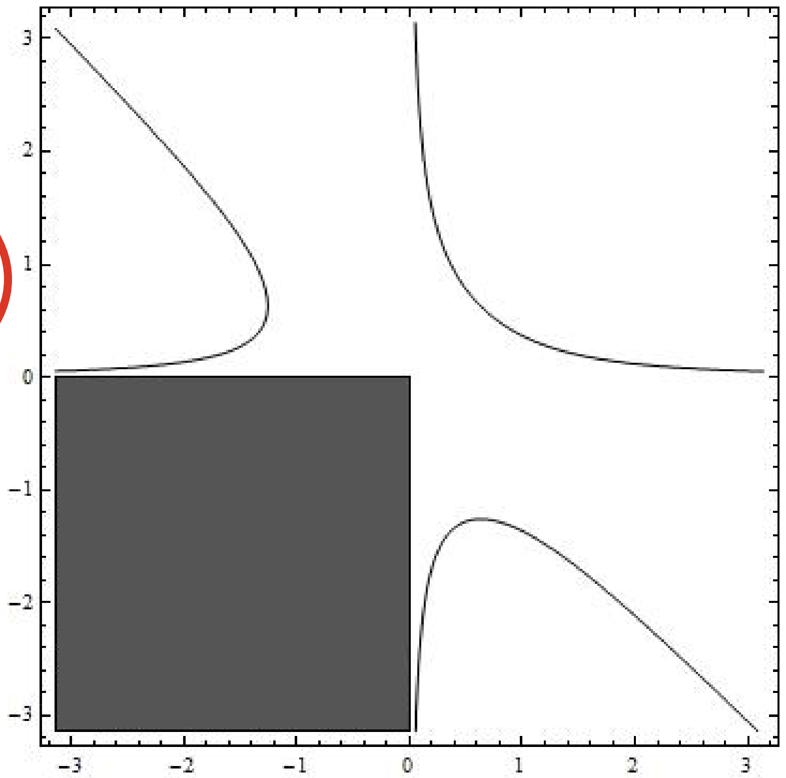
$$x^2y + xy^2 \leq 0.5$$

Interval Arithmetic

Inner Box

$$x \in [-\pi, 0]$$
$$y \in [-\pi, 0]$$

$$[-\pi,0]^2[-\pi,0] + [-\pi,0][-\pi,0]^2 =$$
$$[-2\pi^3, 0] \leq 0.5$$

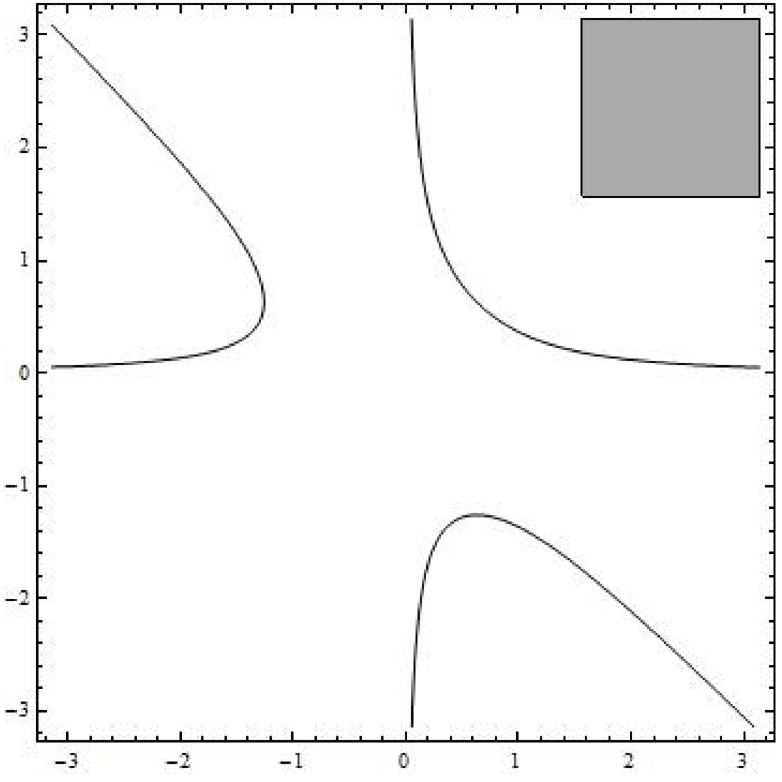


Interval Analysis

$$x^2 y + xy^2 \le 0.5$$

Non-solution Box ?

$$x \in [\pi/2, \pi]$$
$$y \in [\pi/2, \pi]$$



Interval Analysis

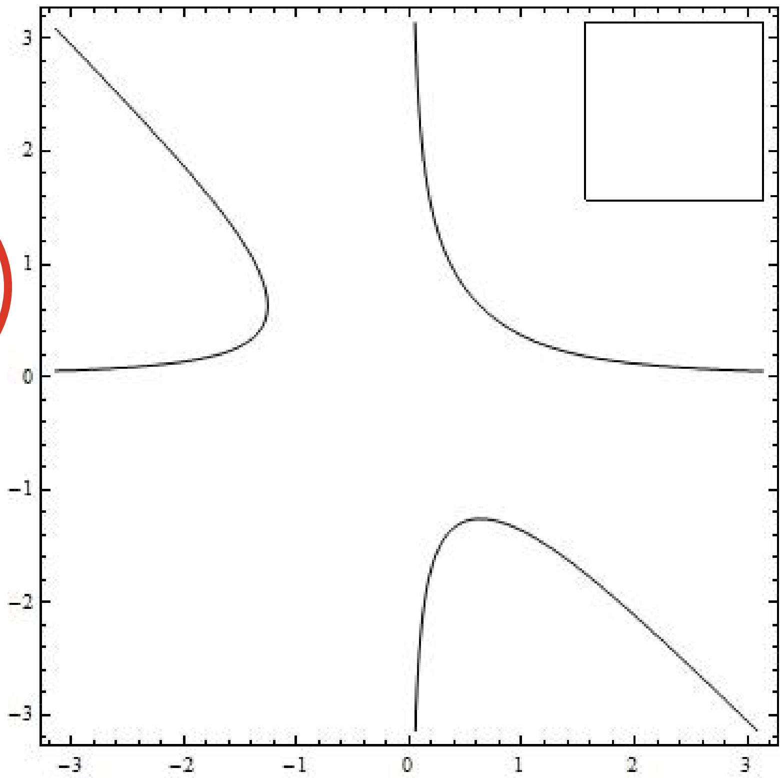
$$x^2y + xy^2 \leq 0.5$$

Interval Arithmetic

Non-solution Box ✓

$$x \in [\pi/2, \pi]$$
$$y \in [\pi/2, \pi]$$

$$[\pi/2,\pi]^2[\pi/2,\pi] + [\pi/2,\pi][\pi/2,\pi]^2 =$$
$$[\pi^3/4, \pi^3] \leq 0.5$$ ✗

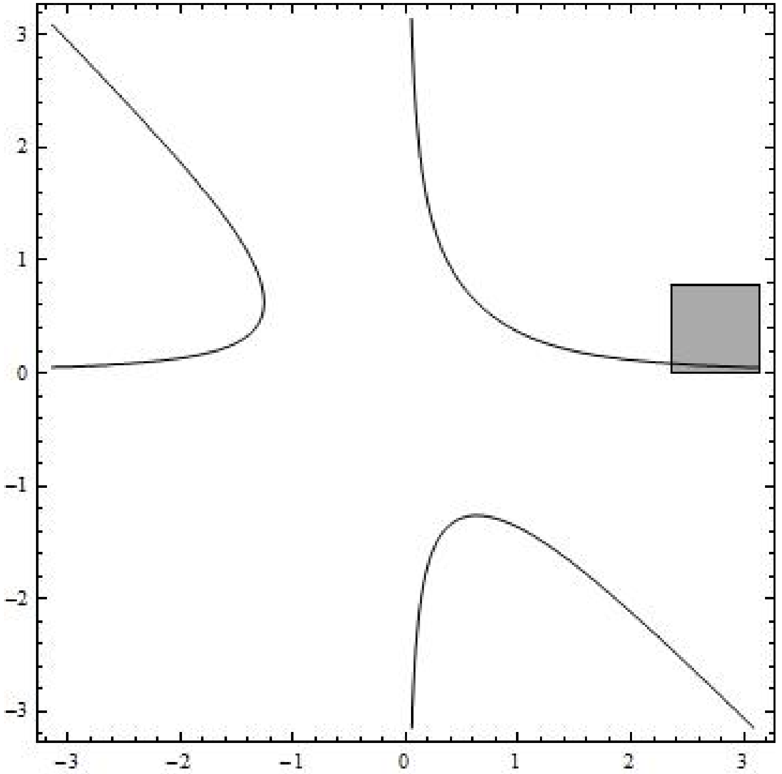


Interval Analysis

$$x^2 y + xy^2 - 0.5 \leq 0$$

Prune Boundary Box

$$[3\pi/4, \pi] \times [0, \pi/4]$$



Interval Analysis

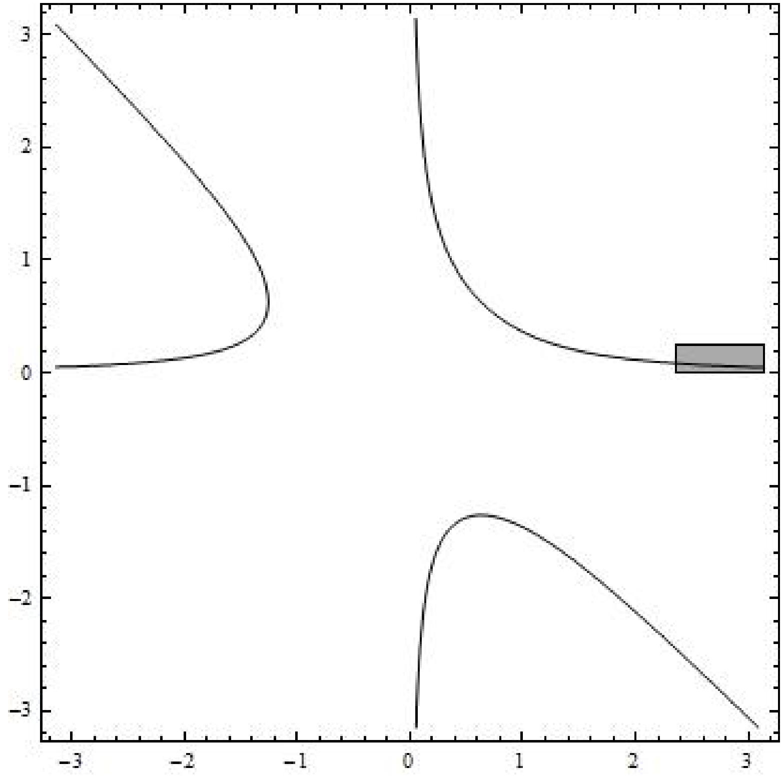
$$x^2 y + xy^2 - 0.5 \le 0$$

Prune Boundary Box

$$[3\pi/4, \pi] \times [0, \pi/4]$$

Newton step

$$[3\pi/4, \pi] \times [0, 0.2547]$$



Interval Analysis

$$x^2 y + xy^2 - 0.5 \leq 0$$

Prune Boundary Box

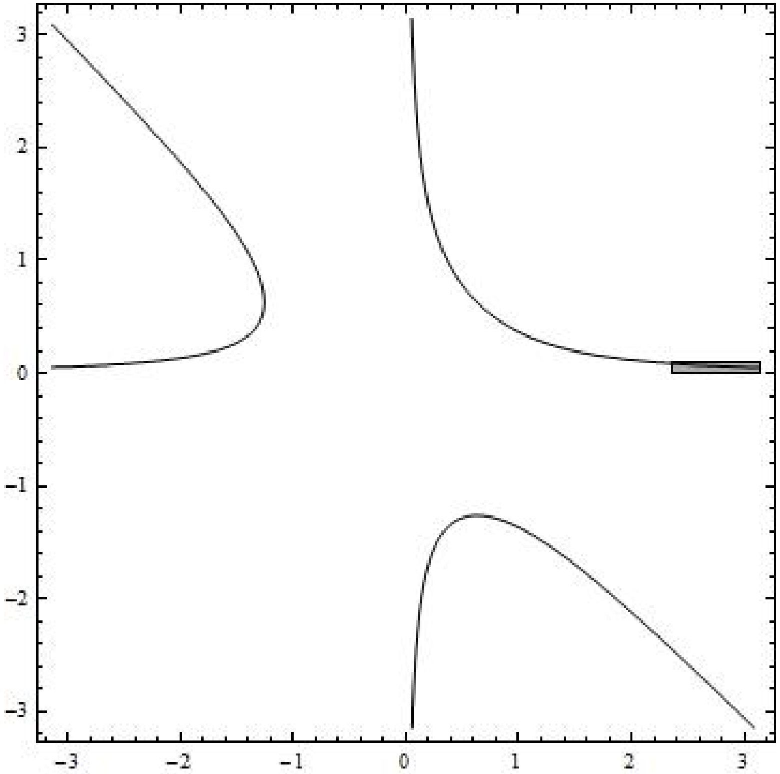
$$[3\pi/4, \pi] \times [0, \pi/4]$$

Newton step

$$[3\pi/4, \pi] \times [0, 0.2547]$$

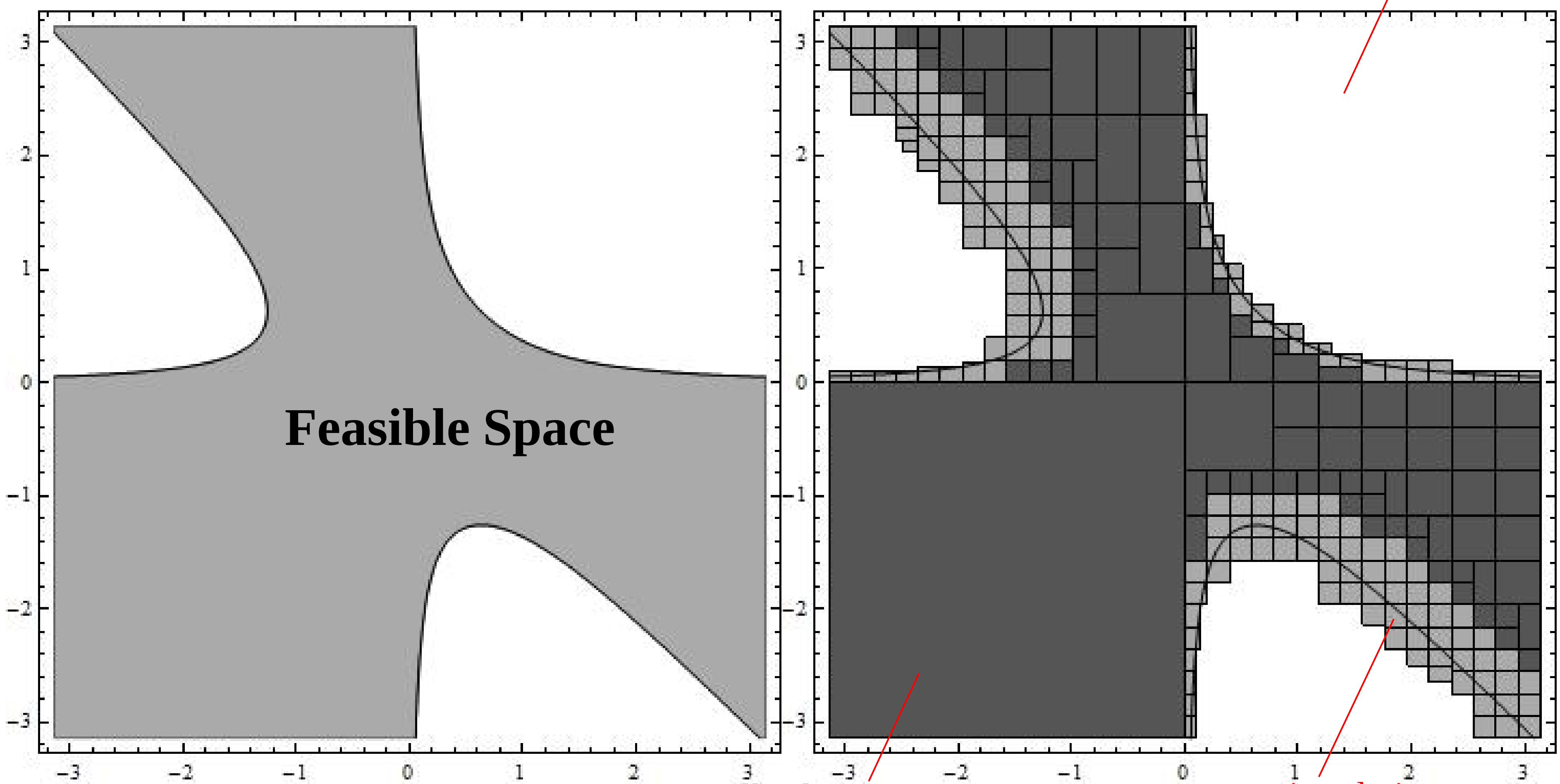
Newton steps

$$[3\pi/4, \pi] \times [0, 0.0874]$$

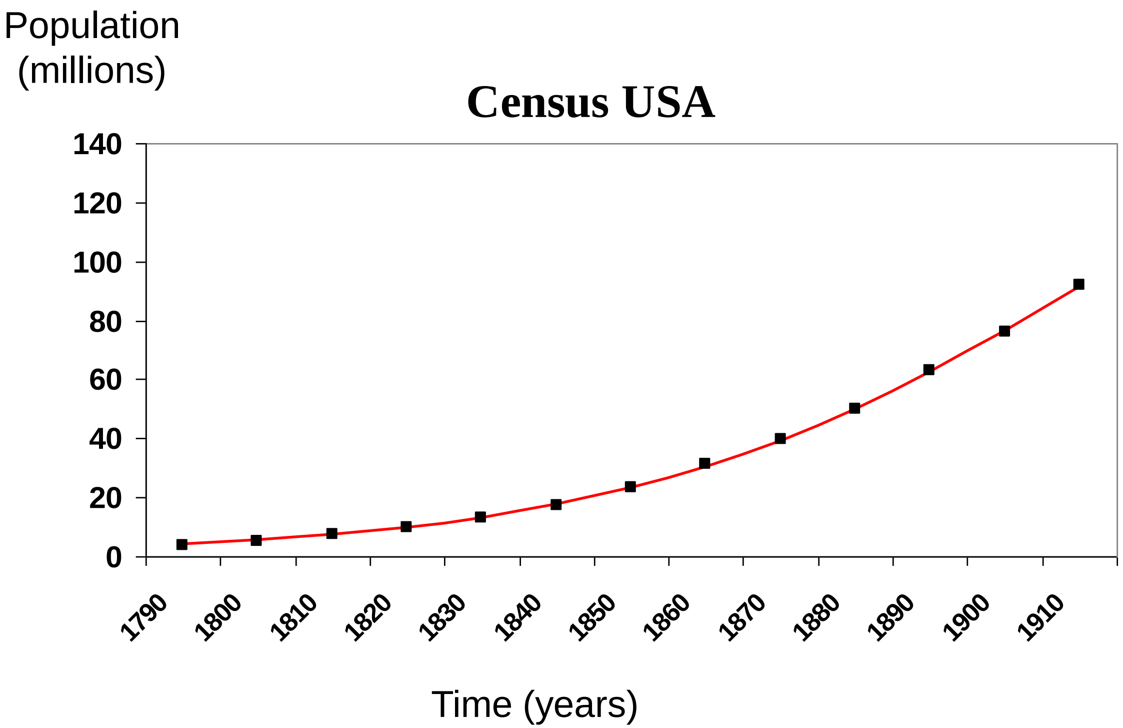


Continuous Constraint Programming

Continuous Constraint Reasoning:



Example: Parameter Estimation



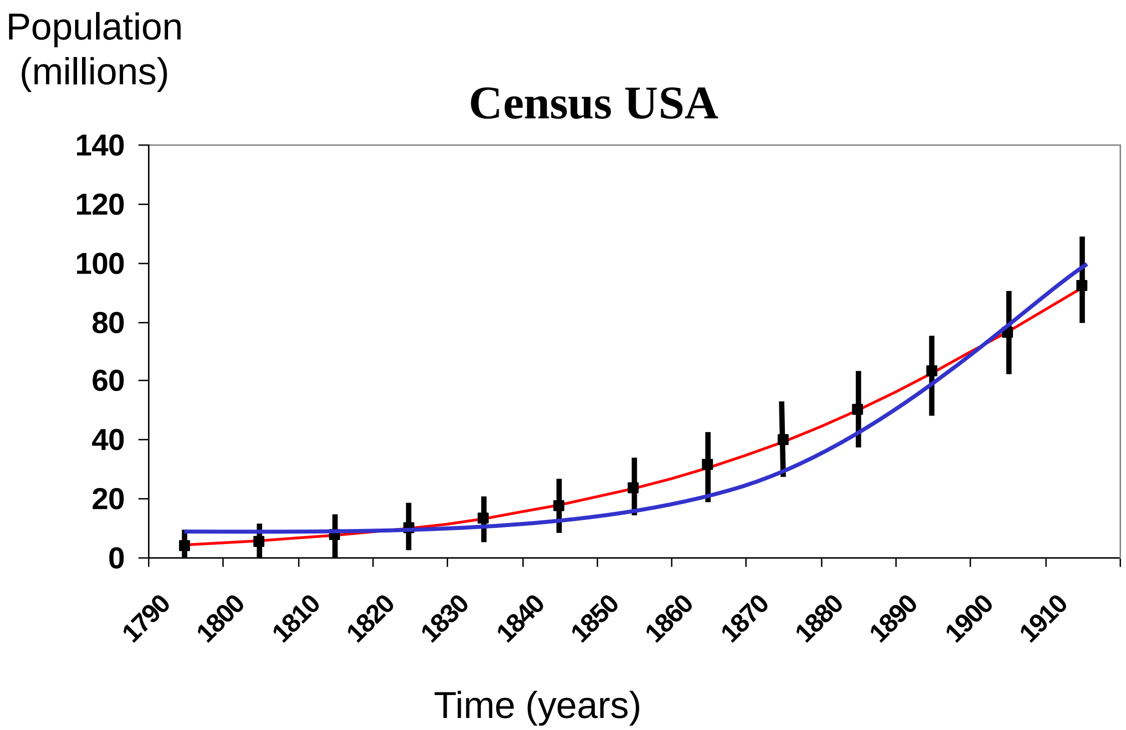
Logistic Model

$$x(t) = \frac{k x_0 e^{r(t-t_0)}}{x_0\left(e^{r(t-t_0)} - 1\right) + k}$$

result: $\begin{cases} x_0 = a \\ k = b \\ r = c \end{cases}$

Optimization Problem: $min \sum_i (x_i - v_i)^2$ with $x_i = \frac{k x_0 e^{r(t_i-t_0)}}{x_0\left(e^{r(t_i-t_0)} - 1\right) + k}$

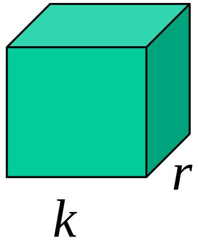
Example: Parameter Estimation



Logistic Model

$$x(t) = \frac{kx_0 e^{r(t-t_0)}}{x_0\left(e^{r(t-t_0)} - 1\right) + k}$$

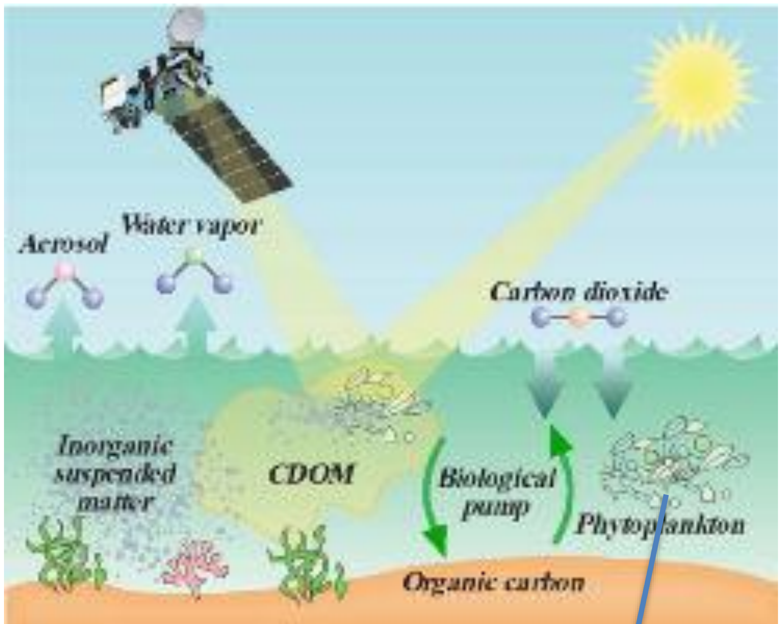
result: x_0



CCSP: $\left\{ \langle x_0, k, r \rangle \mid \forall_{(t_i, v_i)}\ x_i = \frac{kx_0 e^{r(t_i - t_0)}}{x_0\left(e^{r(t_i-t_0)} - 1\right) + k} \quad \wedge \quad |x_i - v_i| \le \varepsilon_i \right\}$

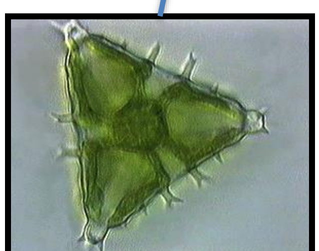
Applications

Ocean Color Remote Sensing



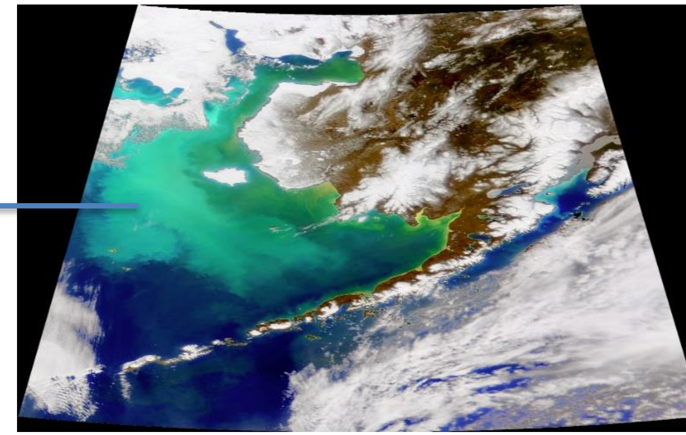
Goal?

phytoplankton concentration

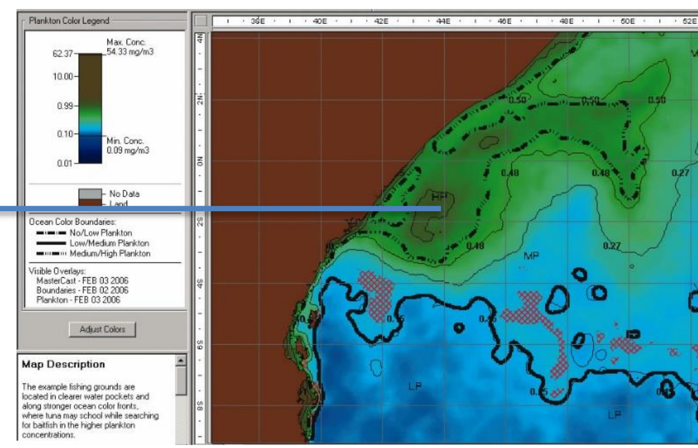


Why measure from space?

harmful algal blooms

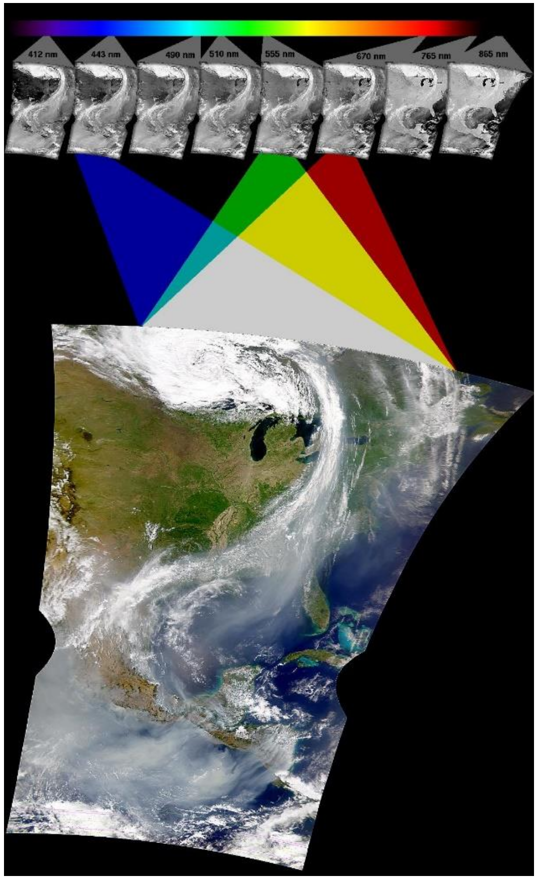


potential fishing zones



Applications

Ocean Color Remote Sensing



Remote Reflectance RR_{λ_i}

errors

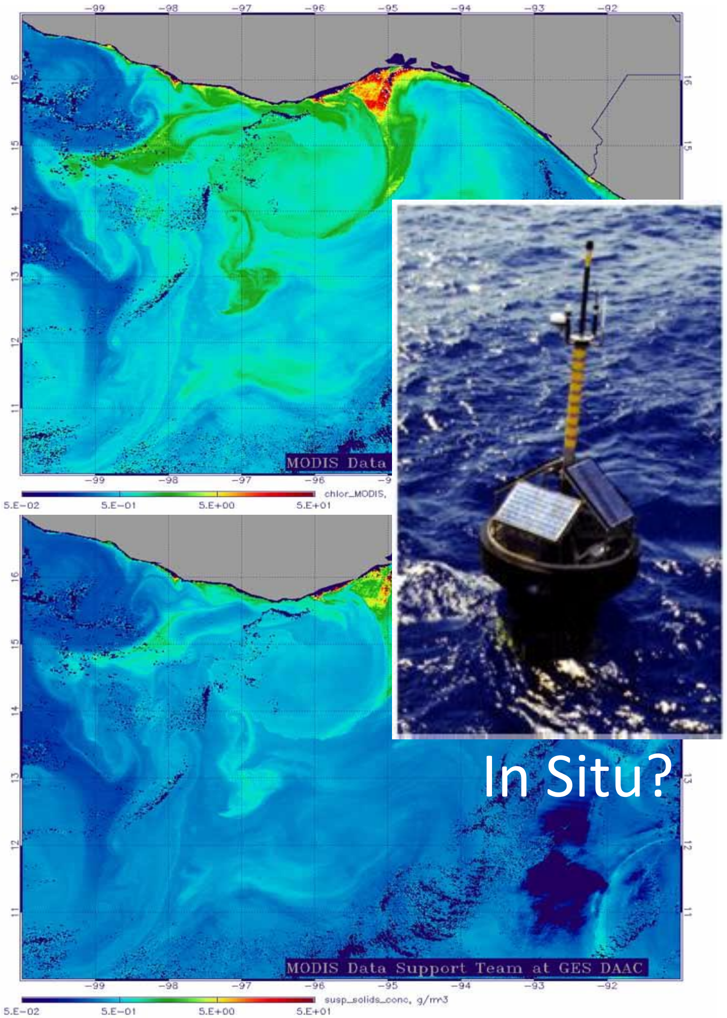
$M^{-1}(\lambda, RR)$

approximation

$M(\lambda, OC)$

nonlinear model

Chlorophyll



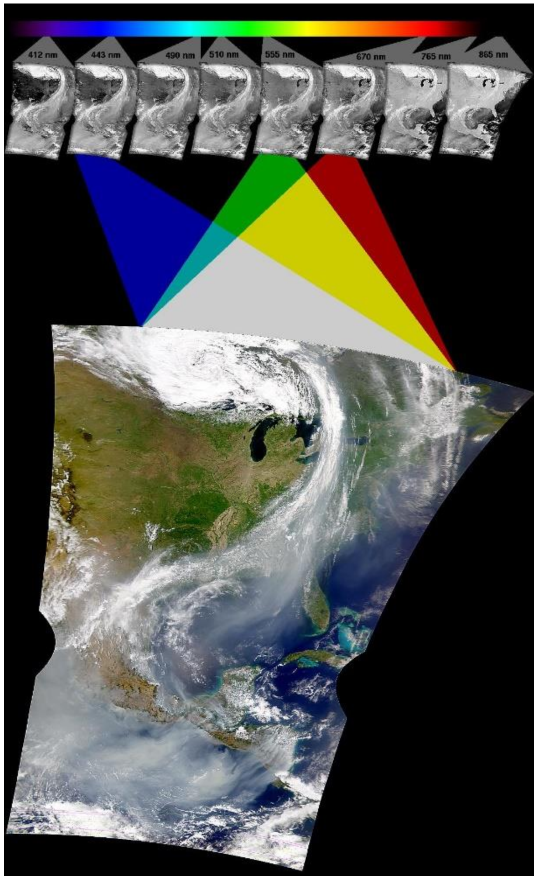
In Situ?

How Accurate?

OC Products OC_j

Applications

Ocean Color Remote Sensing



Remote Reflectance RR_{λ_i}

errors

propagate

$$RR_{\lambda_i} = M(\lambda_i, OC)$$

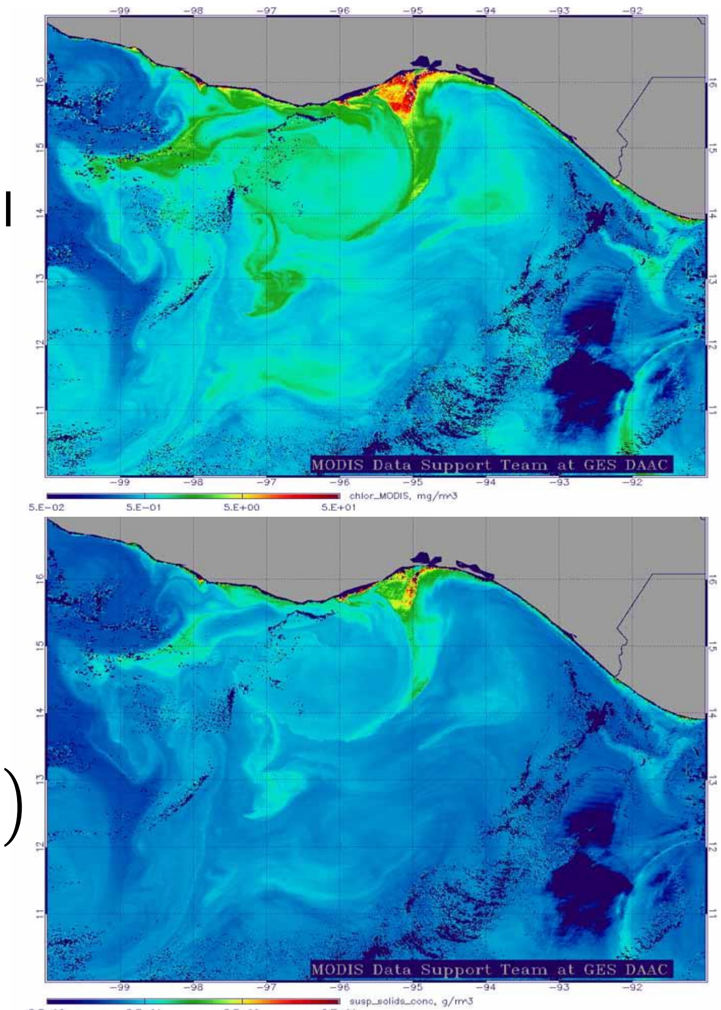
constraint model

probabilistic reasoning

$$P\left(OC | RR_{\lambda_i} = M(\lambda_i, OC)\right)$$

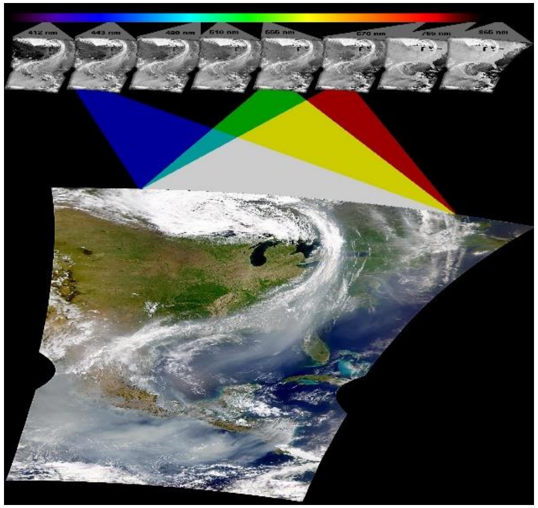
How Accurate?

Chlorophyll



OC Products OC_j

Applications



Remote Reflectance RR_{λ_i}

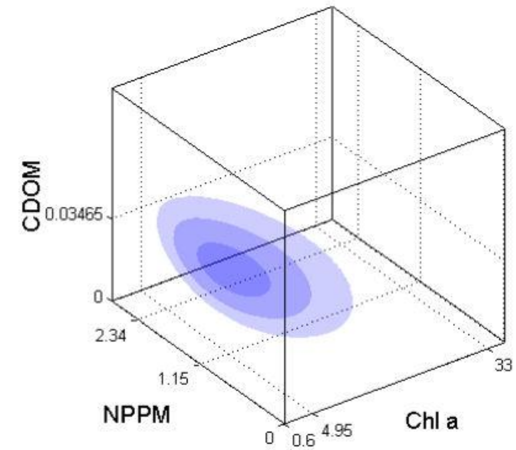
errors

$$RR_{\lambda_i} = M(\lambda_i, OC)$$

constraint model

probabilistic reasoning

$$P\left(OC | RR_{\lambda_i} = M(\lambda_i, OC)\right)$$



OC Products OC_j

measurements accuracy

σ_i	$E[Chla]$		$STD[Chla]$	
	PCTM	PCMC	PCTM	PCMC
100 %	[6.5924, 6.7092]	6.6543	[2.7716, 2.9011]	2.8627
50 %	[5.3576, 5.4262]	5.3932	[1.1822, 1.2412]	1.2266
10 %	[4.9956, 5.0398]	5.0176	[0.2124, 0.2352]	0.2326
5 %	[4.9852, 5.0271]	5.0061	[0.0986, 0.1174]	0.1161
1 %	[4.9827, 5.0224]	5.0025	[0.0103, 0.0235]	0.0232

OC products accuracy

sensor accuracy requirements based on the quality of the estimated OC products

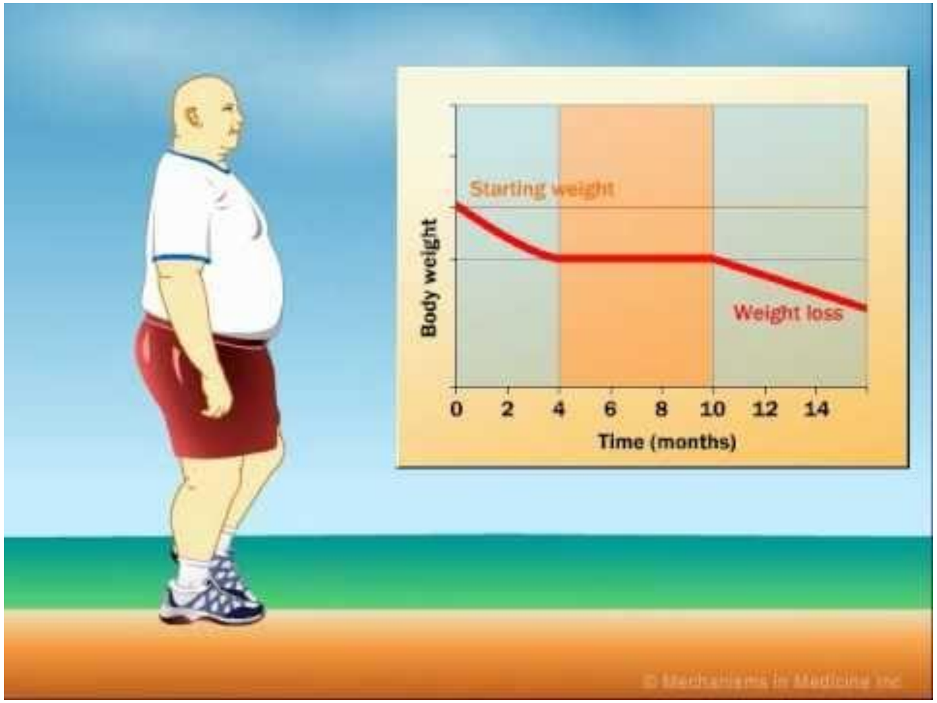
Applications

Biomedical Models

- nonlinear relations
- dynamic behaviour
- uncertainty

Reliable Clinical Tools

- sound propagation of uncertainty from model parameters to results



Monitor energy intake during weight loss treatments

Goal? To offer better/personalized weight loss treatment

Why is a problem? Patients tend to underestimate energy intake
Clinical tests to assess it directly are expensive

Applications

Biomedical Models

Energy Intake Model [Thomas]

$$cf\,\frac{dF}{dt} + cl\,\frac{dFF}{dt} = I - (DIT + PA + RMR + SPA)$$

Differential Equation!

I	Energy intake	
F	Fat mass	
FF	Fat Free mass	$\approx FF'(F, a, h)$
DIT	Diet Induced Thermogenesis	$\approx DIT'(I)$
PA	Physical Activity	$\approx PA'(F)$
RMR	Resting Metabolic Rate	$\approx RMR'(F)$
SPA	Spontaneous Physical Activity	$\approx SPA'(F, I)$

age

height

Nonlinear Constraints!

Fat free mass models:

$$FF^{log}(F) = d_0 + d_1 \log F$$

Uncertainty!

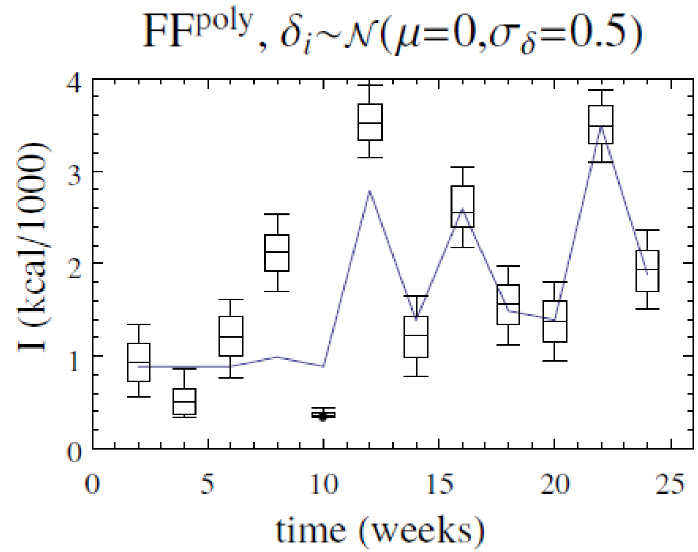
$$FF^{poly}(F, a, h) = (c_0 + c_1 F + c_2 F^2 + c_3 F^3 + c_4 F^4)(c_5 + c_6 a)(c_7 + c_8 h)$$

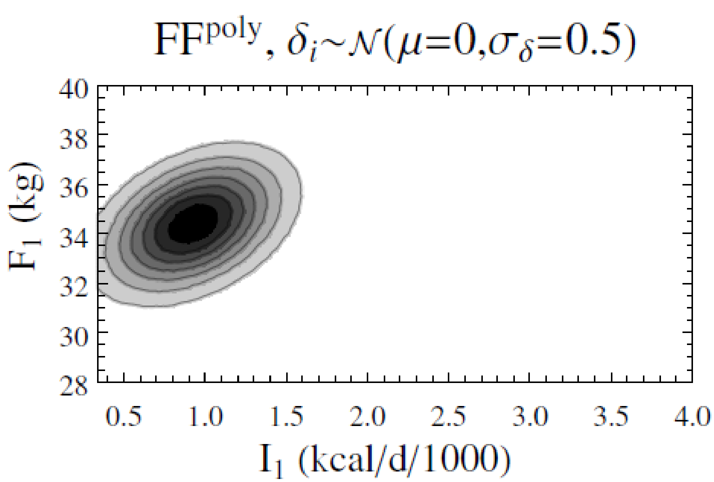
Applications

Energy Intake Model

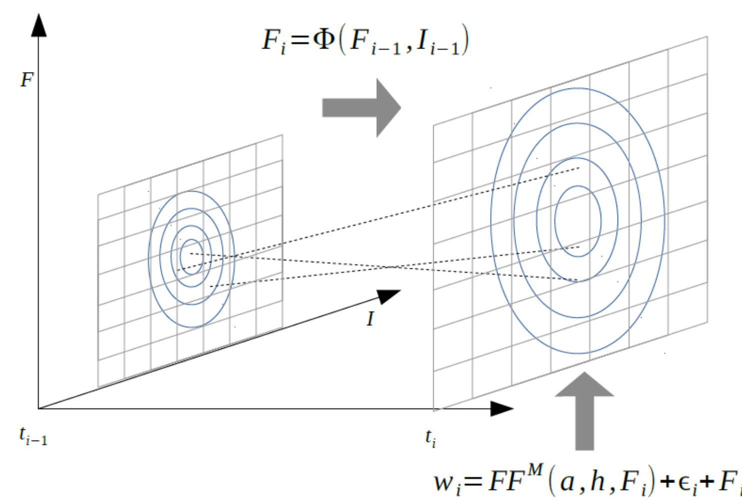
constraint model

probabilistic reasoning



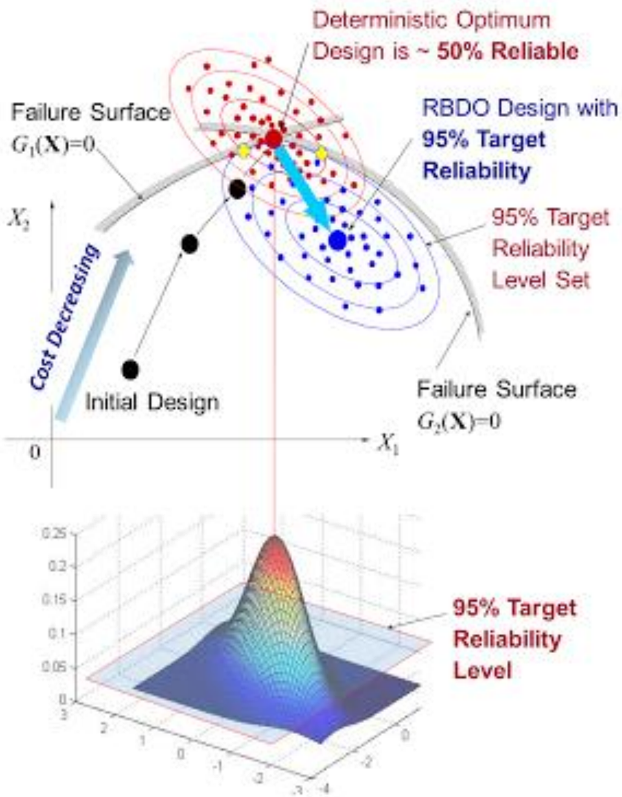


ODE Constraints:

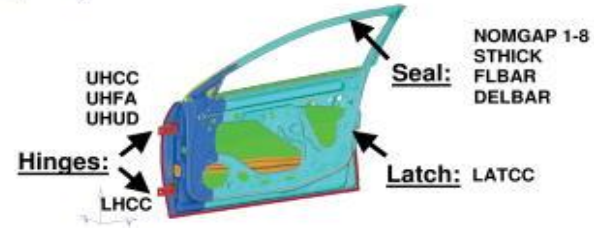


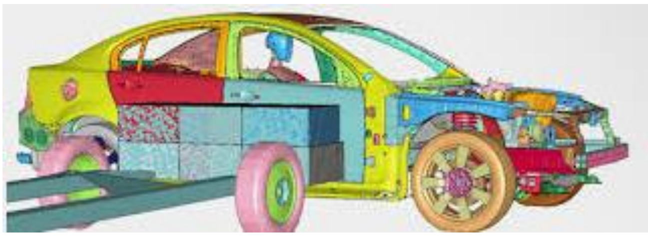
Applications

Reliability Based Design



Crashworthiness of vehicle side impact





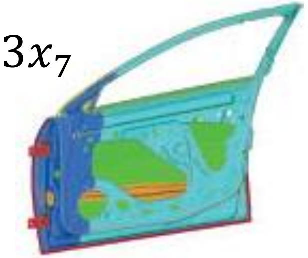
Goal? Design { minimize vehicle weight
reliable to safety requirements

Applications

Reliability Based Design

Crashworthiness of vehicle side impact

weight: $w = 1.98 + 4.90x_1 + 6.67x_2 + 6.98x_3 + 4.01x_4 + 1.78x_5 + 2.73x_7$



Design Variables!

dummy's responses: abdomen load, upper rib deflection, …

Uncertainty!

$r_1 = 1.16 - 0.3717x_2x_4 - 0.00931x_2x_{10} - 0.484x_3x_9 + 0.01343x_6x_{10}$

$r_2 = 28.98 + 3.818x_3 - 4.2x_1x_2 + 0.0207x_5x_{10} + 6.63x_6x_9 - 7.7x_7x_8 + 0.32x_9x_{10}$

$\vdots$

Nonlinear Constraints!

safety requirements: $r_1 \leq 1.0,\ r_2 \leq 0.32, \cdots$

Optimization!

minimize: w

subject to: $P(r_1 \leq 1.0) \geq R$

$P(r_2 \leq 0.32) \geq R$

$\vdots$

linear approximations
individually computed
↓
inconsistent designs!

constraint model
↓ probabilistic reasoning
optimal reliable designs

Applications

constraint model

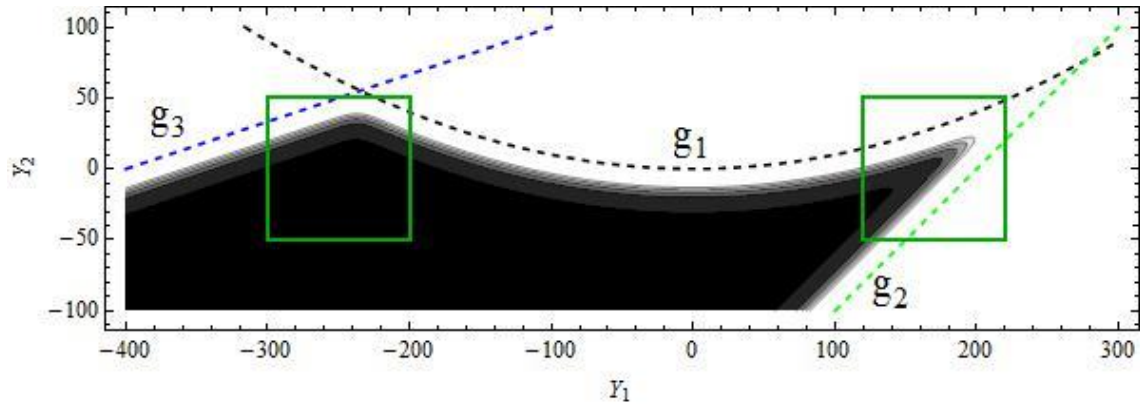
probabilistic reasoning

↓

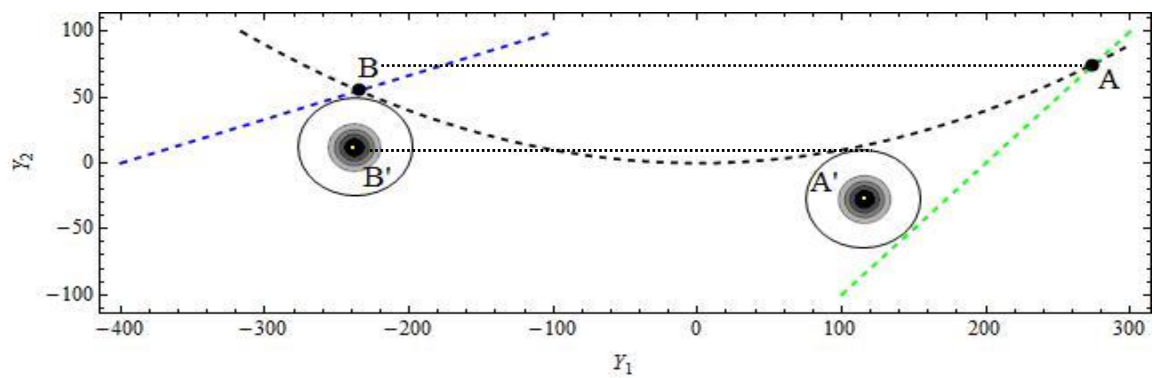
optimal reliable designs

new algorithm:

Reliability distribution of design variables



Reliability based design optimization



Applications

Localization and mapping autonomous robots:
keep track of its current location based on information captured from the environment during its displacement

Model parameters: location of the autonomous robot

Observables: information gathered by a set of robot sensors

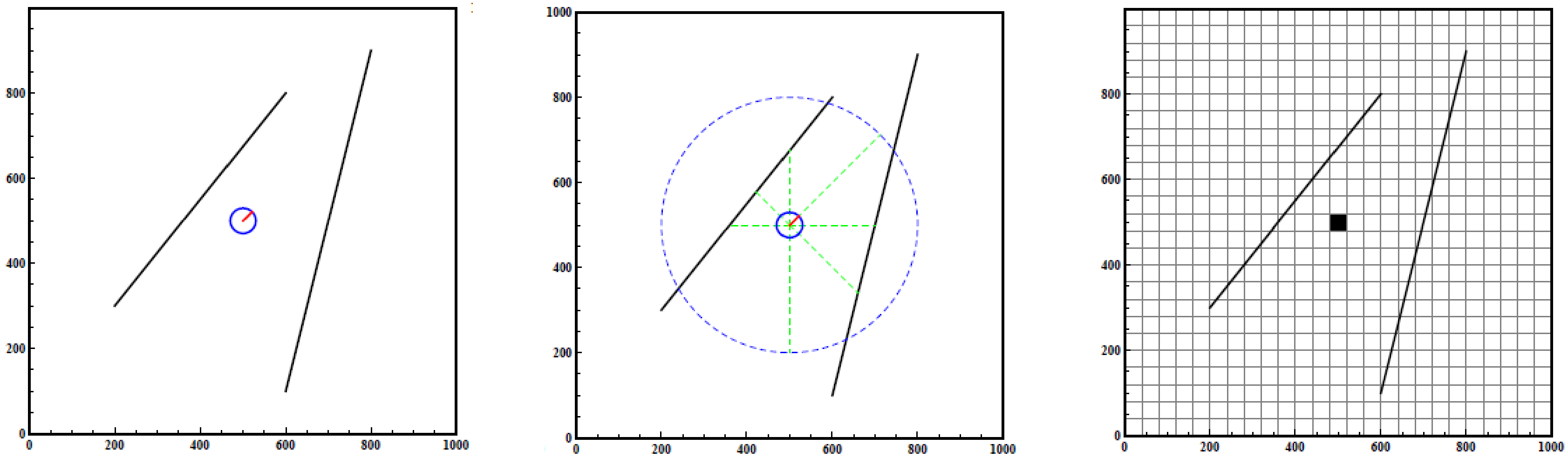
Model: robot kinematic constraints and a priori knowledge about the environment

Measurement errors in accordance with the sensors characteristics

Applications

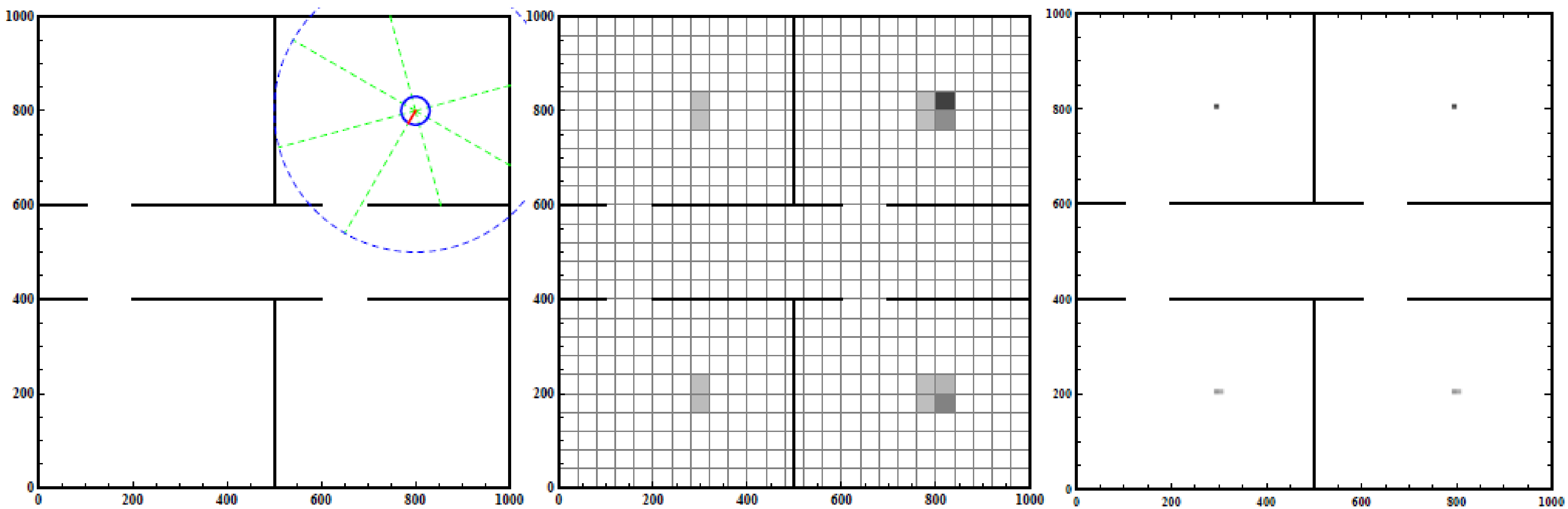
The goal is to determine the robot pose given:

- a set of sensor measurements
- prior knowledge on the map of the environment



Applications

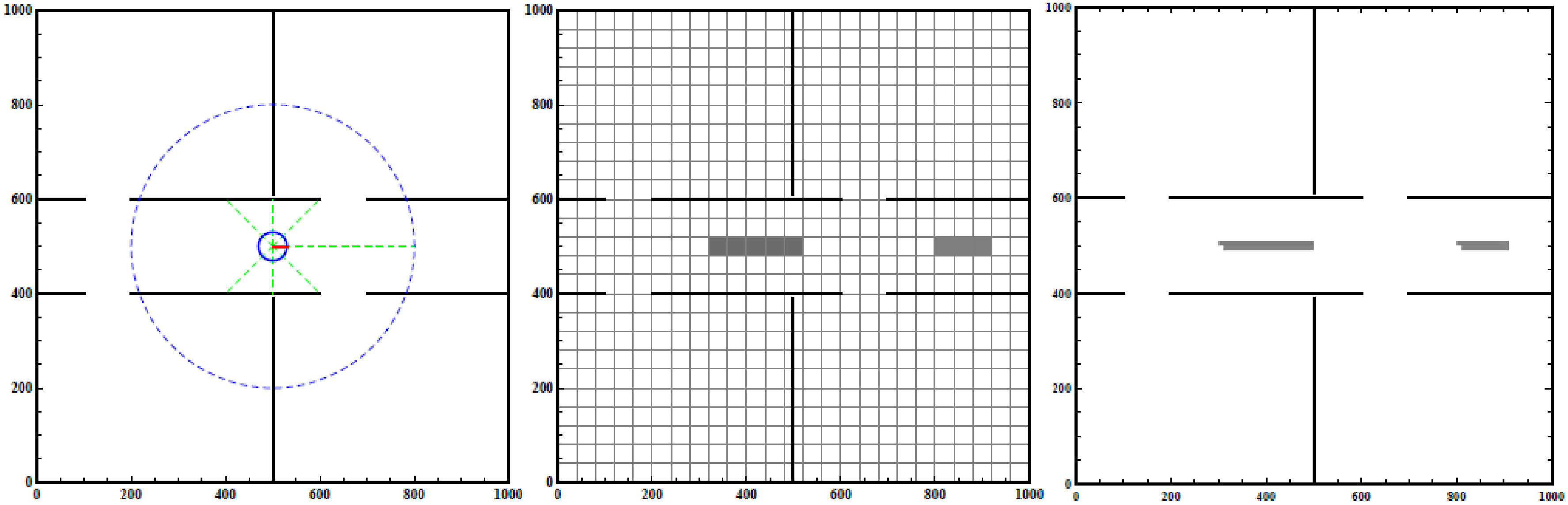
A problem with multiple consistent robot locations



Despite the adopted grid resolution, all the consistent locations are successfully identified.

Applications

A problem with a continuum of consistent robot locations



All the consistent locations were successfully identified

Limitations to represent a continuum of possibilities

Interval Constraint Programming Solvers

Ibex (by Gill Chabert and Luc Jaulin)

A C++ numerical library based on interval arithmetic and constraint programming
http://www.ibex-lib.org/

RealPaver (by L. Granvilliers and F. Benhamou)

A C++ package for modeling and solving nonlinear and nonconvex constraint satisfaction problems
http://pagesperso.lina.univ-nantes.fr/~granvilliers-l/realpaver/

Alias (by Jean-Pierre Merlet, COPRIN team)

A C++ library for system solving, with Maple interface
http://www-sop.inria.fr/coprin/logiciels/ALIAS/ALIAS-C++/ALIAS-C++.html

Interval Constraint Programming Solvers

RSolver (by Stefan Ratschan)

A solver for quantified constraints over the real numbers, implemented in the programming language Ocaml
http://rsolver.sourceforge.net/

Gecode (by Christian Schulte et al)

C++ toolkit for developing constraint-based systems and applications
https://www.gecode.org/

ICOS (by Lebbah)

A software package for rigorously solving global optimization problems
https://www.swmath.org/software/4007

Course Structure: Constraints on Continuous Domains

Lecture 1: Interval Constraints Overview

Lecture 2: Intervals, Interval Arithmetic and Interval Functions

Lecture 3: Interval Newton Method

Lecture 4: Associating Narrowing Functions to Constraints

Lecture 5: Constraint Propagation and Consistency Enforcement

Lecture 6: Constraint Solving

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- Jaulin, L., Kieffer, M., Didrit, O., Walter, E. *Applied Interval Analysis* Springer 2001

Important Links

- ***Interval Computations***
  A primary entry point to items concerning interval computations.
- ***COCONUT - COntinuous CONstraints Updating the Technology***
  Project to integrate techniques from mathematical programming, constraint programming, and interval analysis.

Papers (classics)

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- G. Chabert and L. Jaulin. Hull Consistency under Monotonicity, In Principles and Practice of Constraint Programming - CP 2009, Springer, 188–195, 2009.
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Papers (advanced techniques)

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Papers (generalizations)

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Papers (applications)

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