Constraint Programming

2020/2021 – Mini-Test #2

Monday, 11 January, 9:00 h in 204-Ed.II

Duration: 1.5 h (open book)

1. Interval Arithmetic

Consider the univariate polynomial function expressed in the standard form as:

$$f(x) = x^3 - 7x - 6$$

1.1. Define the mean value extension of f over the interval $[-3/2, -1/2]$ centered at the midpoint.

$F_c(x) = f(c) + F'([a,b]) \times (x - c)$

$[a,b] = [-3/2, -1/2]$

$$c = \frac{-3/2 - 1/2}{2} = -1$$

$f(c) = c^3 - 7c - 6 = (-1)^3 - 7(-1) - 6 = -1 + 7 - 6 = 0$

$F'([a,b]) = 3[a,b]^2 - 7 = 3[-3/2, -1/2]^2 - 7 = 3[1/4, 9/4] - 7 = [-25/4, -1/4]$

$F_c(x) = [-25/4, -1/4] \times (x + 1)$

1.2. Let $I = [-1 - w, -1 + w]$ with $w = 1/2$. Compute enclosures for the range of $f(I)$ with:

a. the standard form;

b. the centered form defined in 1.1

a. standard form:

$f([-3/2, -1/2]) = [-3/2, -1/2]^3 - 7[-3/2, -1/2] - 6$

$= [-27/8, -1/8] - [-21/2, -7/2] - 6$

$= [-27/8, -1/8] + [-5/2, 9/2]$

$= [-47/8, 35/8]$

$= [-5.875, 4.375]$

b. centered form:

$f([-3/2, -1/2]) = [-25/4, -1/4] \times ([-3/2, -1/2] + 1)$

$= [-25/4, -1/4] \times [-1/2, 1/2]$

$= [-25/8, 25/8]$

$= [-3.125, 3.125]$

1.3. Prove that for any positive $w \leq 1/2$ the enclosure for the range of $f([-1 - w, -1 + w])$ obtained with the centered form is sharper that the obtained with the standard form.

a. standard form: $width\big(f([-1 - w, -1 + w])\big)$

$= width([-1 - w, -1 + w]^3) + width(-7[-1 - w, -1 + w]) + 0$

$= width([-1 - w, -1 + w]^3) + width([7 - 7w, 7 + 7w])$

$= width([-1 - w, -1 + w]^3) + 14w > 14w$

b. centered form: $width\big(f([-1 - w, -1 + w])\big)$

$= width([-25/4, -1/4] \times ([-1 - w, -1 + w] + 1))$

$= width([-25/4, -1/4] \times ([-w, +w]))$

$= width([-25w/4, 25w/4]) = 50w/4 = 12.5w$

$\therefore$ for any positive $w \leq 1/2$: width centered form = 12.5w < 14w < width standard form

1.4. Define an algorithm that based on the monotonicity of f computes a sharp enclosure of the range of the function for any interval $[a,b]$.

$f'(x) = 3x^2 - 7$

the roots of the derivative are: $-\sqrt{7/3}$ and $+\sqrt{7/3}$

Algorithm that returns a sharp enclosure of $f([a,b])$:

$I \leftarrow f([a]) \uplus f([b])$

if $\left(a < -\sqrt{7/3} < b\right)$: $I \leftarrow I \uplus f\left(\left[-\sqrt{7/3}\right]\right)$

if $\left(a < +\sqrt{7/3} < b\right)$: $I \leftarrow I \uplus f\left(\left[+\sqrt{7/3}\right]\right)$

return I

2. Interval Newton

Consider the polynomial of the previous question: $f(x) = x^3 - 7x - 6$

2.1. Define the interval Newton function for the polynomial.

$$N([a,b]) = c - \frac{f(c)}{F'([a,b])} \qquad with \quad c = \frac{a+b}{2}$$

$f(c) = c^3 - 7c - 6$

$F'([a,b]) = 3[a,b]^2 - 7$

$$\therefore N([a,b]) = c - \frac{c^3-7c-6}{3[a,b]^2-7} \qquad with \quad c = \frac{a+b}{2}$$

2.2. Use the interval Newton method to compute an interval enclosure of the smallest root of the polynomial within $[-3,0]$. The enclosure must be certified (proved that contains a root) and sharp (width cannot exceed 0.05).

The procedure starts with the initial interval and successively applies the newton function to narrow the leftmost interval that may contain a root. It stops when it proves that an interval smaller that 0.05 contains a root.

All the roots within the initial interval $[-3,0]$ must be in $[-3,0] \cap N([-3,0])$

$$N([-3,0]) = -\frac{3}{2} - \frac{-\frac{27}{8} + \frac{21}{2} - 6}{3[0,9] - 7} = -\frac{3}{2} - \frac{\frac{9}{8}}{[-7,20]} = -\frac{3}{2} - \left(\left[-\infty, -\frac{9}{56}\right] \cup \left[\frac{9}{160}, +\infty\right]\right)$$

$$= \left(\left[-\frac{75}{56}, +\infty\right] \cup \left[-\infty, -\frac{249}{160}\right]\right) = [-\infty, -1.55625] \cup [-1.33929, +\infty]$$

$\therefore$ if there are roots in $[-3,0]$ they must be in:

$$[-3,0] \cap ([-\infty, -1.55625] \cup [-1.33929, +\infty]) = [-3, -1.55625] \cup [-1.33929, 0]$$

Now the leftmost interval $[-3, -1.55625]$ is choosen and the procedure is repeated:

$$N([-3, -1.55625]) = -2.278125 - \frac{-1.87626}{3[2.42191, 9] - 7} = -2.278125 - \frac{-1.87626}{[0.265742, 20]}$$

$$= -2.278125 - [-7.060457, -0.093813] = [-2.184312, 4.78233]$$

$\therefore$ if there are roots in $[-3, -1.55625]$ they must be in:

$$[-3, -1.55625] \cap [-2.184312, 4.78233] = [-2.184312, -1.55625]$$

(it is proved that there are no roots smaller than -2.184312)

Applying the procedure to the interval $[-2.184312, -1.55625]$:

$$N([-2.184312, -1.55625]) = -1.87028 - \frac{0.549816}{3[2.42191, 4.77122] - 7}$$

$$= -1.87028 - \frac{0.549816}{[0.265742, 7.31366]}$$

$$= -1.87028 - [0.0751766, 2.06898] = [-3.93926, -1.94546]$$

$\therefore$ if there are roots in $[-2.184312, -1.55625]$ they must be in:

$$[-2.184312, -1.55625] \cap [-3.93926, -1.94546] = [-2.184312, -1.94546]$$

Applying the procedure to the interval $[-2.184312, -1.94546]$:

$$N([-2.184312, -1.94546]) = -2.06489 - \frac{-0.349964}{3[3.78481, 4.77122] - 7}$$
$$= -2.06489 - \frac{-0.349964}{[4.35444, 7.31366]}$$
$$= -2.06489 - [-0.0803695, -0.0478508]$$
$$= [-2.01704, -1.98452]$$

$\therefore$ if there are roots in $[-2.184312, -1.94546]$ they must be in:

$$[-2.184312, -1.94546] \cap [-2.01704, -1.98452] = [-2.01704, -1.98452]$$

(it is proved that there are no roots smaller than -2.01704)

It is proved that $[-2.01704, -1.98452]$ contains a root since:

$$N([-2.184312, -1.94546]) = [-2.01704, -1.98452] \subset [-2.184312, -1.94546]$$

$[-2.01704, -1.98452]$ is an enclosure of the leftmost root in [-3,0] since the newton method discarded [-3,-2.01704]

$[-2.01704, -1.98452]$ has width 0.03252 < 0.05

3. Constraint Propagation

Consider the constraint $yx^2 + xy^2 = 0.75$ and a box $B = [-1,1] \times [-1,1]$

3.1. Is the constraint box-consistent in box B?

box-consistent in $[-1,1] \times [-1,1]$

$\Leftrightarrow 0 \in [-1,1](-1)^2 + (-1)[-1,1]^2 - 0.75 \wedge 0 \in [-1,1](1)^2 + (1)[-1,1]^2 - 0.75$

$\wedge\, 0 \in (-1)[-1,1]^2 + [-1,1](-1)^2 - 0.75 \wedge 0 \in (1)[-1,1]^2 + [-1,1](1)^2 - 0.75$

$\Leftrightarrow 0 \in [-1,1] + [-1,0] - 0.75 = [-2.75, 0.25] \wedge 0 \in [-1,1] + [0,1] - 0.75 = [-1.75, 1.25]$

$\wedge\, 0 \in [-1,0] + [-1,1] - 0.75 = [-2.75, 0.25] \wedge 0 \in [0,1] + [-1,1] - 0.75 = [-1.75, 1.25]$

since all the resulting intervals include 0, the constraint is box-consistent in box B

3.2. Is the constraint hull-consistent in box B?

hull-consistent in $[-1,1] \times [-1,1]$

$\Leftrightarrow \exists_{y\in[-1,1]} y(-1)^2 + (-1)y^2 - 0.75 = 0 \wedge \exists_{y\in[-1,1]} y(1)^2 + (1)y^2 - 0.75 = 0$

$\wedge\, \exists_{x\in[-1,1]} (-1)x^2 + x(-1)^2 - 0.75 = 0 \wedge \exists_{x\in[-1,1]} (1)x^2 + x(1)^2 - 0.75$

However, equation $y(-1)^2 + (-1)y^2 - 0.75 = 0$ has no real solutions:

$$y(-1)^2 + (-1)y^2 - 0.75 = 0 \Leftrightarrow y^2 - y + 0.75 = 0$$
$$\Leftrightarrow y = \frac{1 \pm \sqrt{1 - 4 \times 0.75}}{2} = \frac{1 \pm \sqrt{-2}}{2}$$

$\therefore$ the constraint is not hull-consistent in box B

3.3. Compute the box B' obtained by applying HC4-revise on the constraint with the initial box B.

HC4-revise enforce hull-consistency on a constraint by implicitly decomposing it into primitive constraints. Since box-consistency is stronger than hull-consistency applied on the primitive constraints obtained by decomposition, and the constraint is box-consistent in box B, then B cannot be narrowed by the HC4-revise. Thus $B' = B = [-1,1] \times [-1,1]$.