Associating Narrowing Functions to Constraints

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Associating Narrowing Functions to Constraints

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Complementary Approaches

Projection Function and its Enclosure

A set of narrowing functions is associated with a constraint by considering projections with respect to each variable in the scope

A projection function identifies from a box:

all the possible values of a particular variable for which there is a value combination belonging to the constraint relation

Projection Function. Let $P=(X,D,C)$ be a CCSP. The projection function with respect to a constraint $c=(s,\rho)\in C$ and a variable $x_i\in s$, denoted $\pi_{x_i}^{\rho}$, obtains a set of real values from a real box B and is defined by:

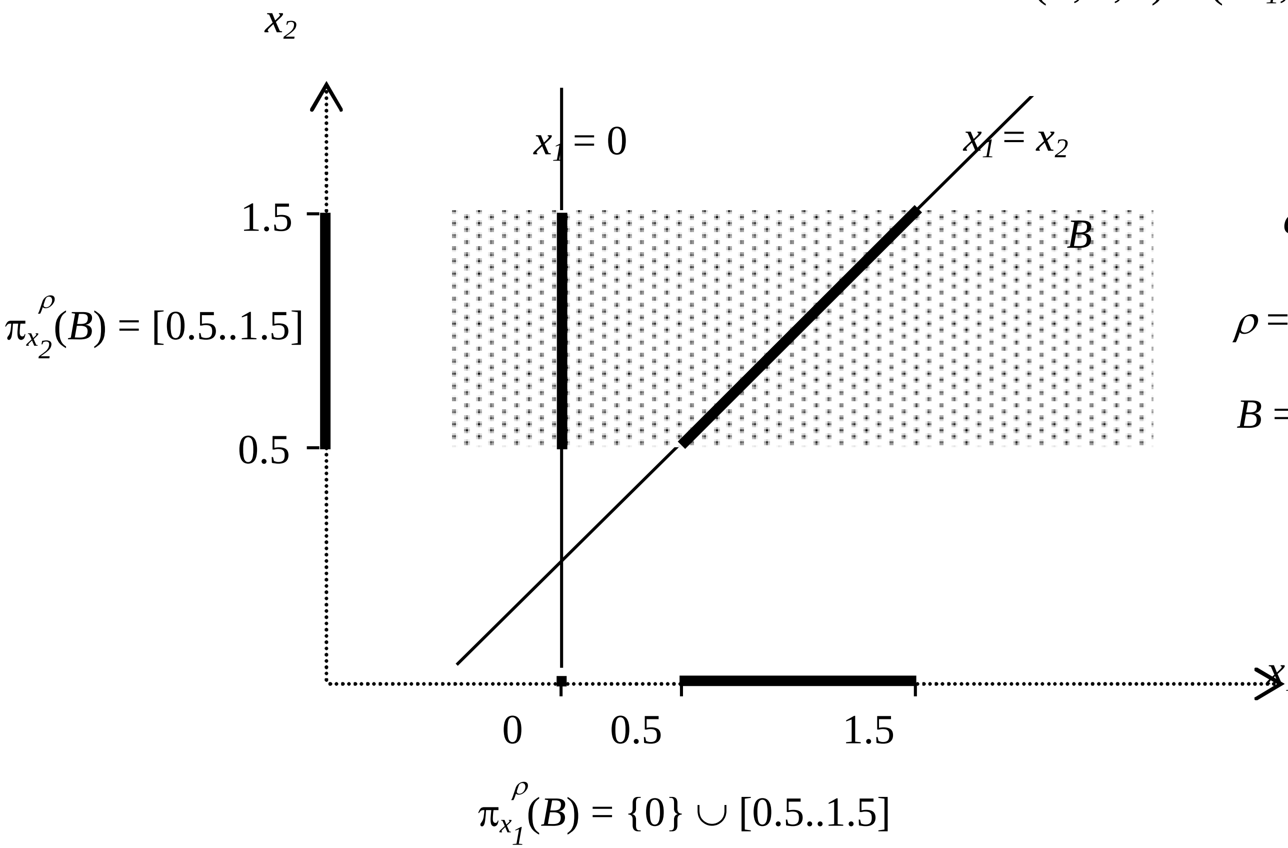
$$\pi_{x_i}^{\rho}(B) = \{\ d[x_i] \mid d \in \rho \wedge d \in B\ \} = (\rho \cap B)[x_i]$$ ❑

All value combinations within B with x_i values outside $\pi_{x_i}^{\rho}(B)$ are outside the relation ρ and so they do not satisfy the constraint c.

Projection Function and its Enclosure

$P = (X,D,C) = (<x_1,x_2>, D_1 \times D_2, \{c\})$

$c \equiv x_1 \times (x_2 - x_1) = 0$



$c = (<x_1,x_2>, \rho)$

$\rho = \{ <x_1,x_2> \in D \mid x_1 \times (x_2 - x_1) = 0 \}$

$B = <[-0.5..2.5],[0.5..1.5]>$

All value combinations within B with x_i values outside $\pi_{x_i}^{\rho}(B)$ are outside the relation ρ and so they do not satisfy the constraint c.

Projection Function and its Enclosure

A box-narrowing function narrows the domain of one variable, from a box representing all the variables of the CCSP, eliminating some values that do not belong to a projection function

Box-Narrowing Function. Let P=(X,D,C) be a CCSP (with X=<x_1,…,x_i,…,x_n>). A box-narrowing function with respect to a constraint $(s,\rho)\in C$ and a variable $x_i \in s$ is a mapping, denoted $\mathrm{BNF}_{x_i}^{\rho}$, that relates any F-box B=<I_{x_1},…, I_{x_i},…, I_{x_n}> ($B \subseteq D$) with the union of m ($1 \leq m$) F-boxes, defined by:

$$\mathrm{BNF}_{x_i}^{\rho}(<I_{x_1},\ldots, I_{x_i},\ldots, I_{x_n}>) = <I_{x_1},\ldots, I_1,\ldots, I_{x_n}> \cup \ldots \cup <I_{x_1},\ldots, I_m,\ldots, I_{x_n}>$$

satisfying the property:

$$\pi_{x_i}^{\rho}(B[s]) \subseteq I_1 \cup \ldots \cup I_m \subseteq I_{x_i}$$ ❑

Contractance follows from $I_1 \cup \ldots \cup I_m \subseteq I_{x_i}$ (the only changed domain is smaller than the original)

Correctness follows from $\pi_{x_i}^{\rho}(B[s]) \subseteq I_1 \cup \ldots \cup I_m$ (the eliminated combinations have x_i values outside the projection function)

Constraint Decomposition Method

Decomposition of complex constraints into an equivalent set of primitive constraints whose projection functions can be easily computed by inverse functions

Primitive Constraints

Primitive Constraint. Let e_c be a real expression with at most one basic operator and with no multiple occurrences of its variables. Let e_0 be a real constant or a real variable not appearing in e_c. The constraint c is a primitive constraint iff it is expressed as:

$$e_c \diamond e_0 \quad \text{with} \quad \diamond \in \{\leq, =, \geq\}$$ ❑

A set of primitive constraints can be easily obtained from any non-primitive constraint:

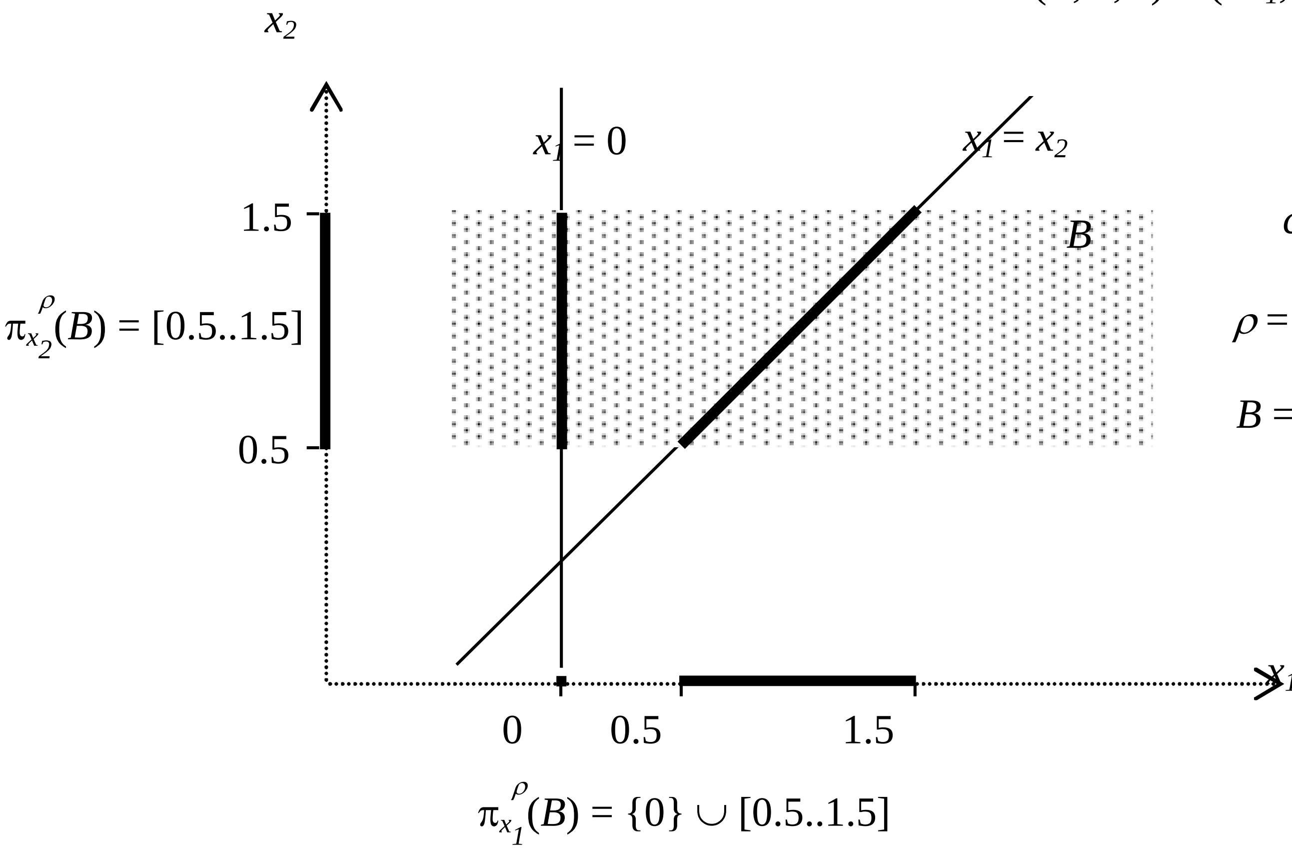
- A constraint may be decomposed by considering new variables and new equality constraints
- The whole set of primitives may be obtained by repeating this process until all constraints are primitive

Constraint Decomposition Method

Primitive Constraints

$P = (X,D,C) = (\langle x_1,x_2\rangle, D_1 \times D_2, \{c\})$

$c \equiv x_1 \times (x_2 - x_1) = 0$



$c = (\langle x_1,x_2\rangle, \rho)$

$\rho = \{ \langle x_1,x_2\rangle \in D \mid x_1 \times (x_2 - x_1) = 0 \}$

$B = \langle [-0.5..2.5],[0.5..1.5] \rangle$

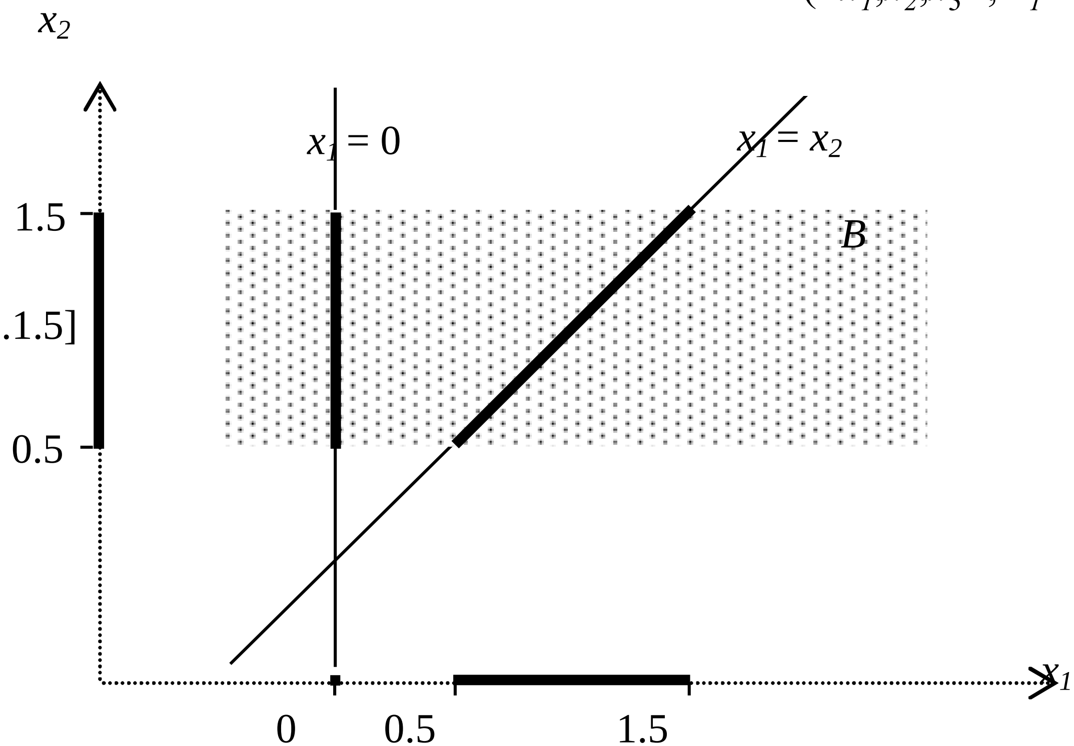
The constraint c is not primitive since it contains two basic arithmetic operators and the variable x_1 occurs twice

Constraint Decomposition Method

Primitive Constraints

$P' = (\langle x_1, x_2, x_3 \rangle, D_1 \times D_2 \times [-\infty..+\infty], \{c_1, c_2\})$

$c \begin{cases} c_1 \equiv x_1 \times x_3 = 0 \\ c_2 \equiv x_2 - x_1 = x_3 \end{cases}$



$\pi^{\rho}_{x_2}(B) = [0.5..1.5]$

$\pi^{\rho}_{x_1}(B) = \{0\} \cup [0.5..1.5]$

a new variable x_3 is introduced and c is replaced by c_1 and c_2

the domain of x_3 is unbounded defining a new equivalent CCSP P'

Constraint Decomposition Method

Inverse Functions

Inverse Interval Expression. Let $c=(s,\rho)$ be a primitive constraint expressed in the form $e_c \diamond e_0$ where $e_c \equiv e_1$ or $e_c \equiv \Phi(e_1,\ldots,e_m)$ (Φ is an m-ary basic operator and e_i a variable from s or a real constant). The inverse interval expression of c with respect to e_i, denoted Ψe_i, is the natural interval expression of the expression obtained by solving algebraically, wrt e_i, the equality $e_c=e_0$ if c is an equality or $e_c=e_0+k$ if c is an inequality (with $k\leq 0$ for inequalities of the form $e_c \leq e_0$ and $k\geq 0$ for inequalities of the form $e_c \geq e_0$). ❑

	Ψe_1	Ψe_2	Ψe_3
$e_1+e_2 \diamond e_3$	$(E_3+K)-E_2$	$(E_3+K)-E_1$	$(E_1+E_2)-K$
$e_1-e_2 \diamond e_3$	$(E_3+K)+E_2$	$E_1-(E_3+K)$	$(E_1-E_2)-K$
$e_1 \times e_2 \diamond e_3$	$(E_3+K)/E_2$	$(E_3+K)/E_1$	$(E_1 \times E_2)-K$
$e_1/e_2 \diamond e_3$	$(E_3+K)\times E_2$	$E_1/(E_3+K)$	$(E_1/E_2)-K$
$e_1 \diamond e_2$	(E_2+K)	E_1-K	

$\diamond \in \{\leq,=,\geq\}$

e_i is a real variable or a real constant

E_i is the natural interval extension of e_i

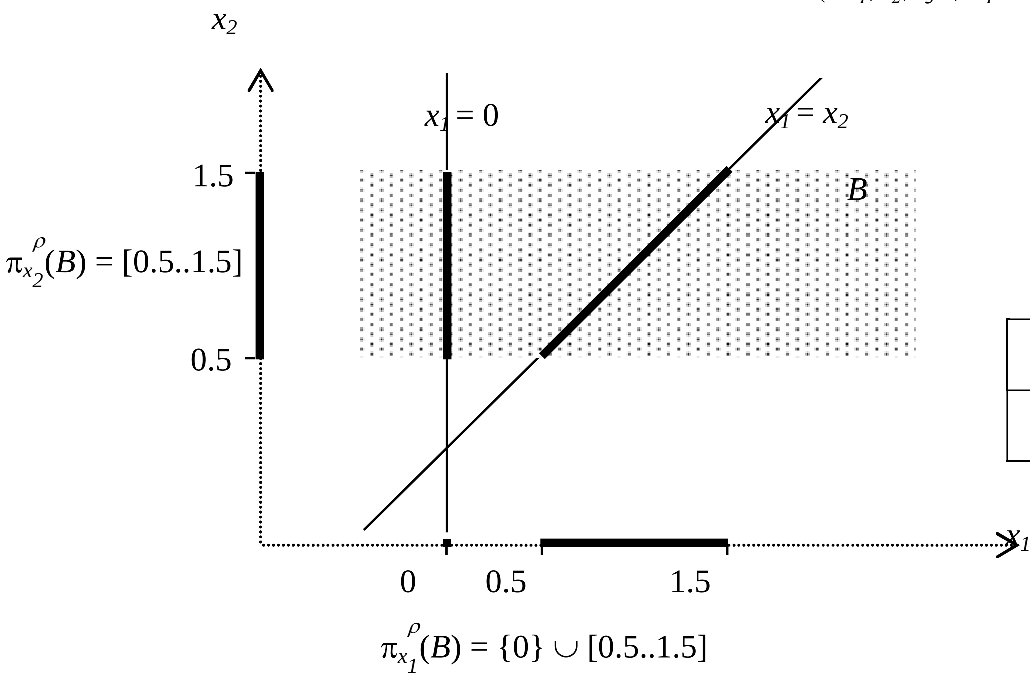
$$K=\begin{cases}[-\infty..0] & \text{if } \diamond \equiv \leq \\ [0..0] & \text{if } \diamond \equiv = \\ [0..+\infty] & \text{if } \diamond \equiv \geq\end{cases}$$

Constraint Decomposition Method

Inverse Functions

$P' = (<x_1, x_2, x_3>, D_1 \times D_2 \times [-\infty..+\infty], \{c_1, c_2\})$

$$c \begin{cases} c_1 \equiv x_1 \times x_3 = 0 \\ c_2 \equiv x_2 - x_1 = x_3 \end{cases}$$



	Ψ_{e_1}	Ψ_{e_2}	Ψ_{e_3}
$x_1 \times x_3 = 0$	$0/X_3$	$0/X_1$	$X_1 \times X_3$
$x_2 - x_1 = x_3$	$X_3 + X_1$	$X_2 - X_3$	$X_2 - X_1$

The inverse interval expressions are associated with the primitive constraints of the decomposed CCSP *P'*

Constraint Decomposition Method

Projection Function Enclosure with the Inverse Function

The inverse interval expression wrt a variable allows the definition of the projection function of the constraint wrt to that variable

Projection Function based on the Inverse Interval Expression. Let $P=(X,D,C)$ be a CCSP. Let $c=(s,\rho)\in C$ be an n-ary primitive constraint expressed in the form $e_c \diamond e_0$ where $e_c \equiv e_1$ or $e_c \equiv \Phi(e_1,\ldots,e_m)$ (with Φ an m-ary basic operator and e_i a variable from s or a real constant). Let Ψx_i be the inverse interval expression of c with respect to the variable x_i ($e_i \equiv x_i$). The projection function $\pi_{x_i}^{\rho}$ of the constraint c wrt variable x_i is the mapping:

$$\pi_{x_i}^{\rho}(B) = \Psi x_i(B) \cap B[x_i] \qquad \text{where } B \text{ is an } n\text{-ary real box}$$ ❑

$x_1 \times x_3 = 0$
$\pi_{x_1}^{\rho}(\langle I_1,I_3\rangle) = (0/I_3) \cap I_1$
$\pi_{x_3}^{\rho}(\langle I_1,I_3\rangle) = (0/I_1) \cap I_3$

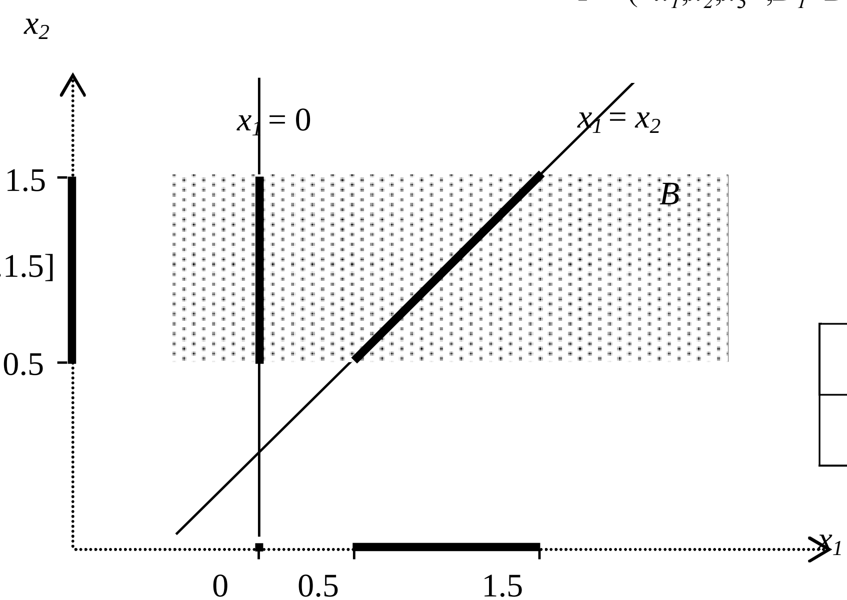
$x_2 - x_1 = x_3$
$\pi_{x_1}^{\rho}(\langle I_1,I_2,I_3\rangle) = (I_2 - I_3) \cap I_1$
$\pi_{x_2}^{\rho}(\langle I_1,I_2,I_3\rangle) = (I_3 + I_1) \cap I_2$
$\pi_{x_3}^{\rho}(\langle I_1,I_2,I_3\rangle) = (I_2 - I_1) \cap I_3$

Constraint Decomposition Method

Projection Function Enclosure with the Inverse Function

$P' = (\langle x_1, x_2, x_3 \rangle, D_1 \times D_2 \times [-\infty..+\infty], \{c_1, c_2\})$

$$c \begin{cases} c_1 \equiv x_1 \times x_3 = 0 \\ c_2 \equiv x_2 - x_1 = x_3 \end{cases}$$



$\pi^{\rho}_{x_2}(B) = [0.5..1.5]$

$\pi^{\rho}_{x_1}(B) = \{0\} \cup [0.5..1.5]$

	Ψ_{e_1}	Ψ_{e_2}	Ψ_{e_3}
$x_1 \times x_3 = 0$	$0/X_3$	$0/X_1$	$X_1 \times X_3$
$x_2 - x_1 = x_3$	$X_3 + X_1$	$X_2 - X_3$	$X_2 - X_1$

Box-narrowing functions are associated with the decomposed CCSP *P'*

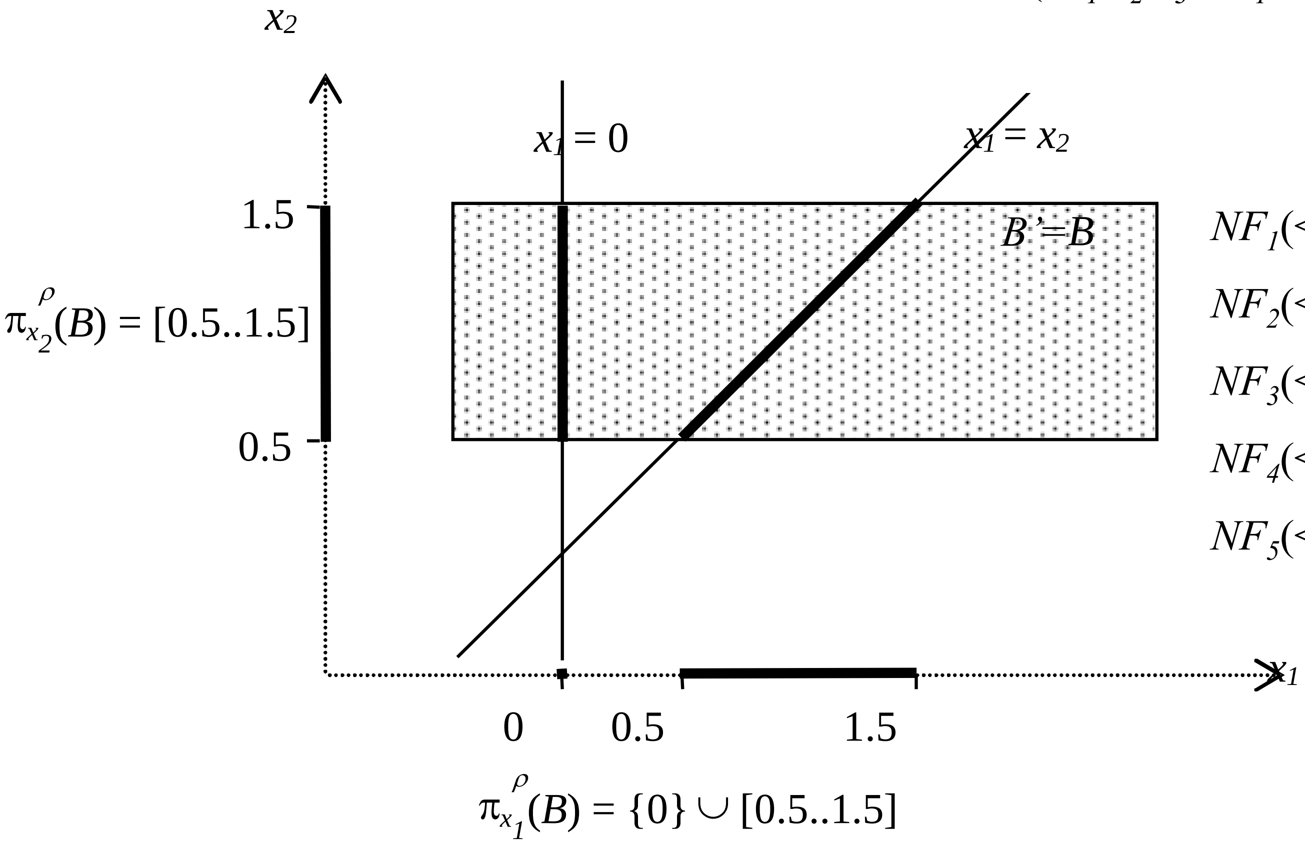
$x_1 \times x_3 = 0$	NF_1	$BNF^{\rho}_{x_1}(\langle I_1, I_2, I_3 \rangle) = \langle (0/I_3) \cap I_1, I_2, I_3 \rangle$
	NF_2	$BNF^{\rho}_{x_3}(\langle I_1, I_2, I_3 \rangle) = \langle I_1, I_2, (0/I_1) \cap I_3 \rangle$
$x_2 - x_1 = x_3$	NF_3	$BNF^{\rho}_{x_1}(\langle I_1, I_2, I_3 \rangle) = \langle (I_2 - I_3) \cap I_1, I_2, I_3 \rangle$
	NF_4	$BNF^{\rho}_{x_2}(\langle I_1, I_2, I_3 \rangle) = \langle I_1, (I_3 + I_1) \cap I_2, I_3 \rangle$
	NF_5	$BNF^{\rho}_{x_3}(\langle I_1, I_2, I_3 \rangle) = \langle I_1, I_2, (I_2 - I_1) \cap I_3 \rangle$

Constraint Decomposition Method

Example

$P' = (\langle x_1, x_2, x_3 \rangle, D_1 \times D_2 \times [-\infty..+\infty], \{c_1, c_2\})$

$$c \begin{cases} c_1 \equiv x_1 \times x_3 = 0 \\ c_2 \equiv x_2 - x_1 = x_3 \end{cases}$$



$NF_1(\langle I_1, I_2, I_3 \rangle) = \langle (0/I_3) \cap I_1, I_2, I_3 \rangle$

$NF_2(\langle I_1, I_2, I_3 \rangle) = \langle I_1, I_2, (0/I_1) \cap I_3 \rangle$

$NF_3(\langle I_1, I_2, I_3 \rangle) = \langle (I_2 - I_3) \cap I_1, I_2, I_3 \rangle$

$NF_4(\langle I_1, I_2, I_3 \rangle) = \langle I_1, (I_3 + I_1) \cap I_2, I_3 \rangle$

$NF_5(\langle I_1, I_2, I_3 \rangle) = \langle I_1, I_2, (I_2 - I_1) \cap I_3 \rangle$

with $B = \langle [-0.5,2.5],[0.5,1.5] \rangle$ no pruning would be obtained:

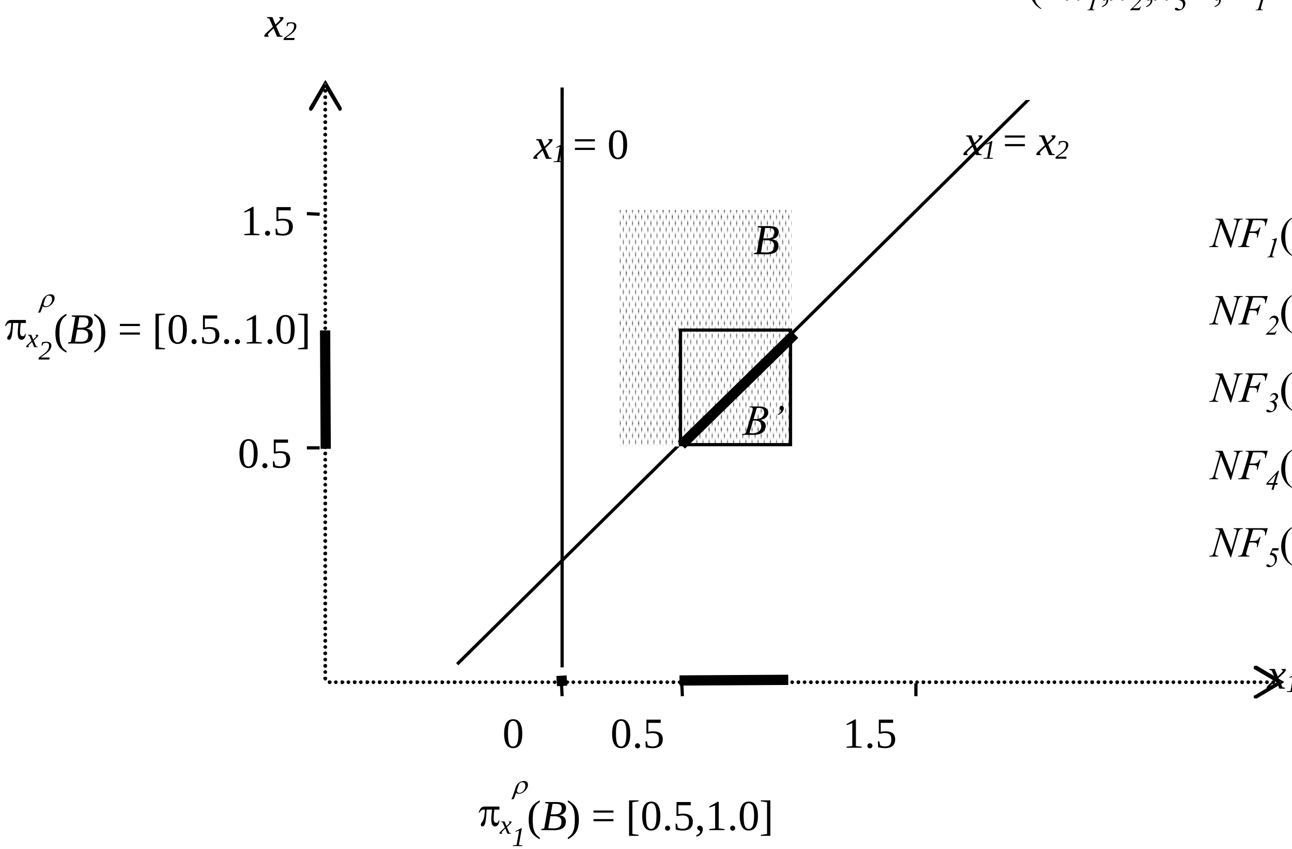
$B' = \langle [-0.5,2.5],[0.5,1.5] \rangle$ and $x_3 = [-2.0,2.0]$

Constraint Decomposition Method

Example

$$P' = (\langle x_1, x_2, x_3 \rangle, D_1 \times D_2 \times [-\infty..+\infty], \{c_1, c_2\})$$

$$c \begin{cases} c_1 \equiv x_1 \times x_3 = 0 \\ c_2 \equiv x_2 - x_1 = x_3 \end{cases}$$



$$NF_1(\langle I_1, I_2, I_3 \rangle) = \langle (0/I_3) \cap I_1, I_2, I_3 \rangle$$

$$NF_2(\langle I_1, I_2, I_3 \rangle) = \langle I_1, I_2, (0/I_1) \cap I_3 \rangle$$

$$NF_3(\langle I_1, I_2, I_3 \rangle) = \langle (I_2 - I_3) \cap I_1, I_2, I_3 \rangle$$

$$NF_4(\langle I_1, I_2, I_3 \rangle) = \langle I_1, (I_3 + I_1) \cap I_2, I_3 \rangle$$

$$NF_5(\langle I_1, I_2, I_3 \rangle) = \langle I_1, I_2, (I_2 - I_1) \cap I_3 \rangle$$

with $B = \langle [0.25,1.0],[0.5,1.5] \rangle$ the best narrowing is obtained:

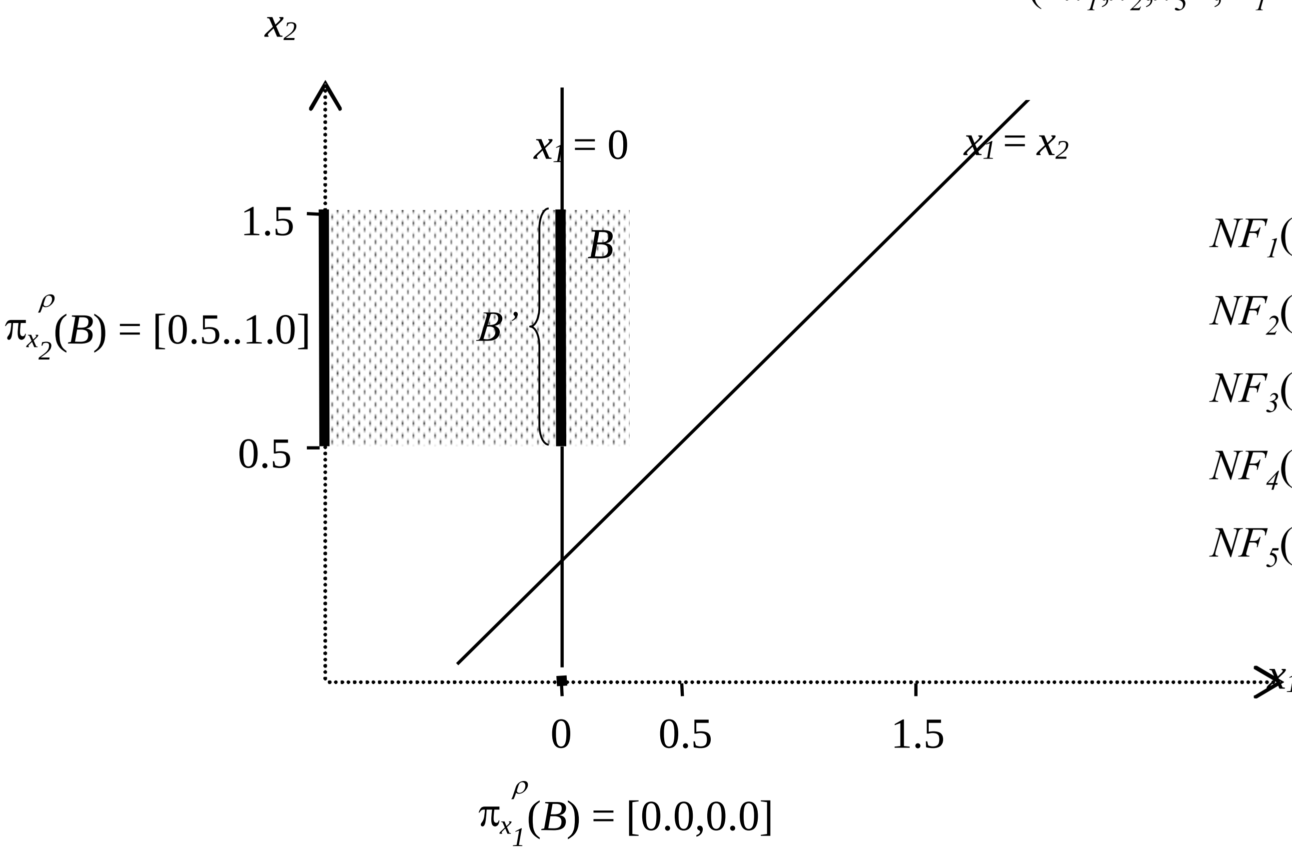
$B' = \langle [0.5,1.0],[0.5,1.0] \rangle$ and $x_3 = [0.0,0.0]$

Constraint Decomposition Method

Example

$P' = (\langle x_1, x_2, x_3 \rangle, D_1 \times D_2 \times [-\infty..+\infty], \{c_1, c_2\})$

$$c \begin{cases} c_1 \equiv x_1 \times x_3 = 0 \\ c_2 \equiv x_2 - x_1 = x_3 \end{cases}$$



$NF_1(\langle I_1, I_2, I_3 \rangle) = \langle (0/I_3) \cap I_1, I_2, I_3 \rangle$

$NF_2(\langle I_1, I_2, I_3 \rangle) = \langle I_1, I_2, (0/I_1) \cap I_3 \rangle$

$NF_3(\langle I_1, I_2, I_3 \rangle) = \langle (I_2 - I_3) \cap I_1, I_2, I_3 \rangle$

$NF_4(\langle I_1, I_2, I_3 \rangle) = \langle I_1, (I_3 + I_1) \cap I_2, I_3 \rangle$

$NF_5(\langle I_1, I_2, I_3 \rangle) = \langle I_1, I_2, (I_2 - I_1) \cap I_3 \rangle$

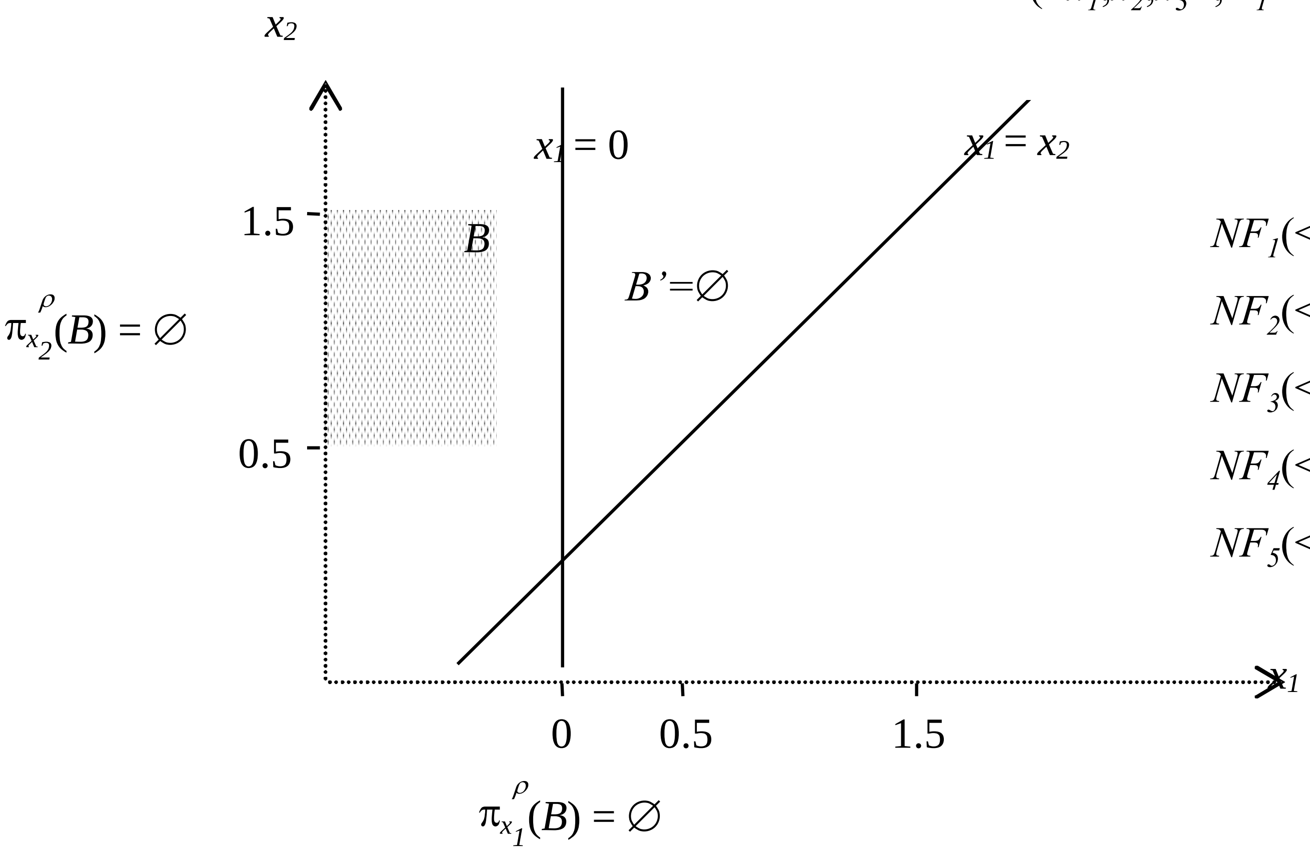
with $B = \langle [-1.0, 0.25], [0.5, 1.5] \rangle$ the best narrowing is also obtained: $B' = \langle [0.0, 0.0], [0.5, 1.5] \rangle$ and $x_3 = [0.5, 1.5]$

Constraint Decomposition Method

Example

$P' = (<x_1, x_2, x_3>, D_1 \times D_2 \times [-\infty..+\infty], \{c_1, c_2\})$

$c \begin{cases} c_1 \equiv x_1 \times x_3 = 0 \\ c_2 \equiv x_2 - x_1 = x_3 \end{cases}$



$NF_1(<I_1,I_2,I_3>) = <(0/I_3) \cap I_1, I_2, I_3>$

$NF_2(<I_1,I_2,I_3>) = <I_1, I_2, (0/I_1) \cap I_3>$

$NF_3(<I_1,I_2,I_3>) = <(I_2-I_3) \cap I_1, I_2, I_3>$

$NF_4(<I_1,I_2,I_3>) = <I_1, (I_3+I_1) \cap I_2, I_3>$

$NF_5(<I_1,I_2,I_3>) = <I_1, I_2, (I_2-I_1) \cap I_3>$

with $B = <[-1.0,-0.25],[0.5,1.5]>$ inconsistency is proved:

$B' = \varnothing$

Constraint Newton Method

Complex constraints are handled without decomposition using a technique based on the interval Newton method for searching the zeros of univariate functions

Interval Projections

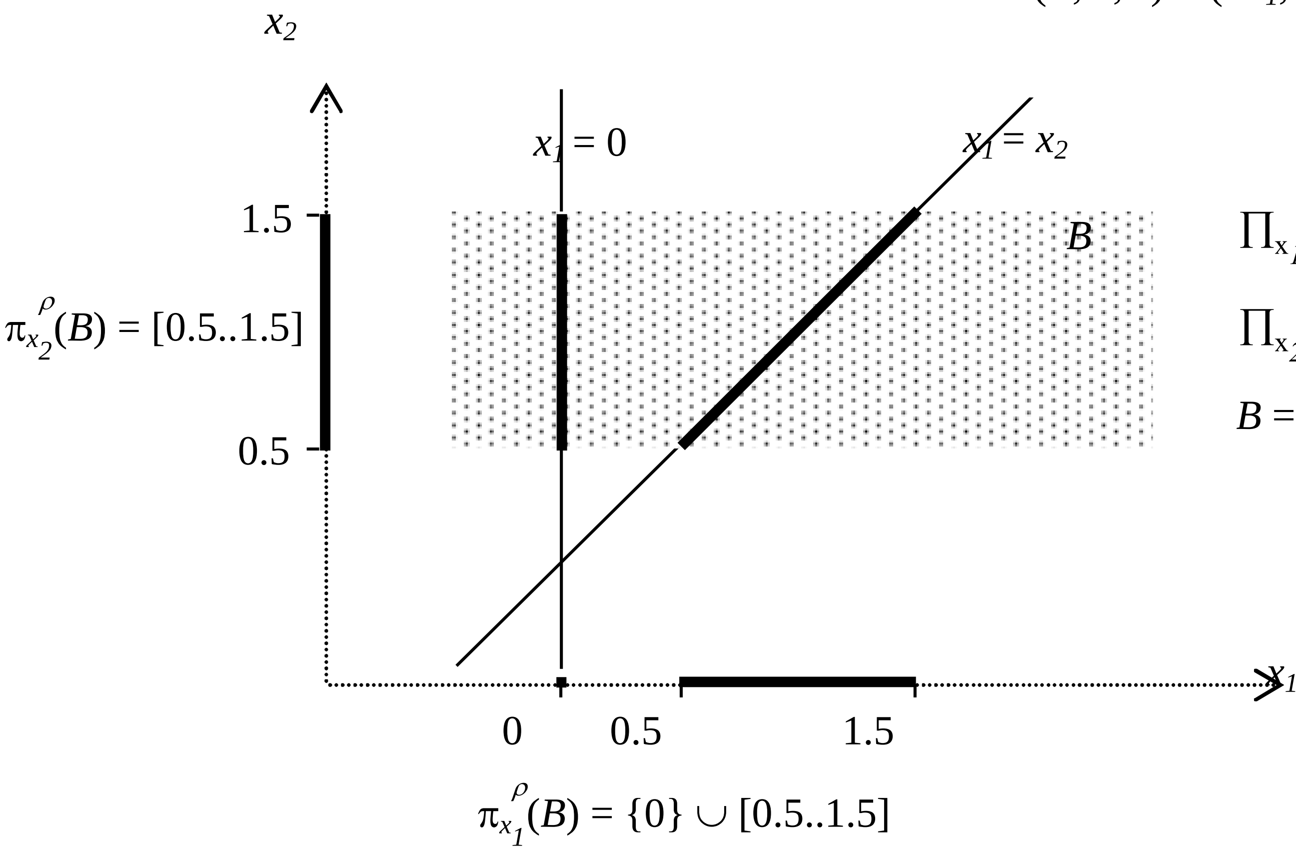
Interval Projection. Let P=(X,D,C) be a CCSP. Let c=(s,ρ)$\in C$ be an n-ary constraint expressed in the form $e_c \diamond 0$ (with $\diamond \in \{\leq,=,\geq\}$ and e_c a real expression). Let B be an n-ary F-box. The interval projection of c wrt $x_i \in s$ and B is the function, denoted $\prod_{x_i}^{\rho B}$, represented by the expression obtained by replacing in e_c each real variable x_j ($x_j \neq x_i$) by the interval constant $B[x_j]$. ❑

Constraint Newton Method

Interval Projections

$P = (X,D,C) = (<x_1,x_2>,D_1 \times D_2,\{c\})$

$c \equiv x_1 \times (x_2 - x_1) = 0$



$\prod_{x_1}^{\rho B} \equiv x_1 \times ([0.5..1.5] - x_1)$

$\prod_{x_2}^{\rho B} \equiv [-0.5..2.5] \times (x_2 - [-0.5..2.5])$

$B = <[-0.5..2.5],[0.5..1.5]>$

All value combinations within B with x_i values outside $\pi_{x_i}^{\rho}(B)$ are outside the relation ρ and so they do not satisfy the constraint c.

Constraint Newton Method

Properties of an Interval Projection

From the properties of the interval projections, a strategy is devised for obtaining an enclosure of the projection function

Properties of the Interval Projection. Let $P=(X,D,C)$ be a CCSP. Let $c=(s,\rho)\in C$ be an n-ary constraint expressed in the form $e_c\diamond 0$ (with $\diamond\in\{\leq,=,\geq\}$ and e_c a real expression) and B an n-ary F-box. Let $\prod_{x_i}^{\rho B}$ be the interval projection of c wrt variable $x_i\in s$ and B. The following property is necessarily satisfied:

$$\forall_{r\in B[x_i]}\ r\in\pi_{x_i}^{\rho}(B)\Rightarrow\exists v\in\prod_{x_i}^{\rho B}([r]):\ v\diamond 0$$

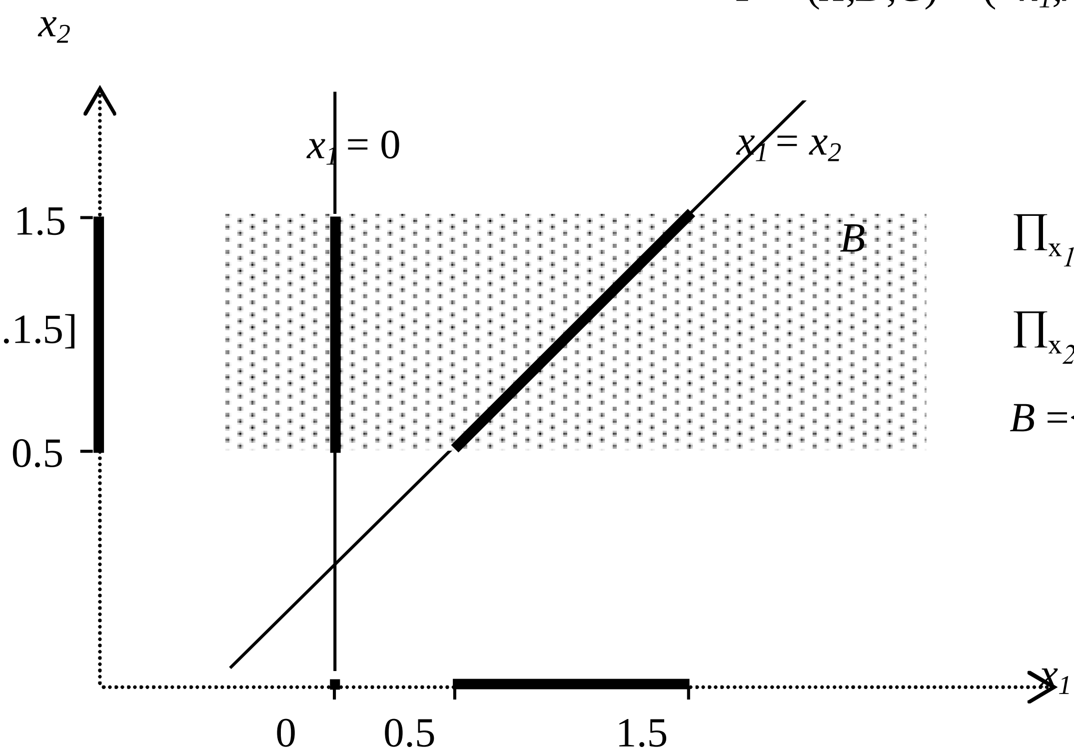
We will say that a real value r satisfies the interval projection condition if the right side of the implication is satisfied. ❑

Constraint Newton Method

Properties of an Interval Projection

$P = (X,D,C) = (<x_1,x_2>,D_1 \times D_2,\{c\})$

$c \equiv x_1 \times (x_2 - x_1) = 0$



$\pi_{x_2}^{\rho}(B) = [0.5..1.5]$

$\pi_{x_1}^{\rho}(B) = \{0\} \cup [0.5..1.5]$

$\prod_{x_1}^{\rho B} \equiv x_1 \times ([0.5..1.5] - x_1)$

$\prod_{x_2}^{\rho B} \equiv [-0.5..2.5] \times (x_2 - [-0.5..2.5])$

$B = <[-0.5..2.5],[0.5..1.5]>$

$\forall_{r \in [-0.5,2.5]} \; r \in \pi_{x_1}^{\rho}(B) \Rightarrow \exists v \in [r] \times ([0.5..1.5] - [r]): \; v = 0$

$\forall_{r \in [0.5,1.5]} \; r \in \pi_{x_2}^{\rho}(B) \Rightarrow \exists v \in [-0.5..2.5] \times ([r] - [-0.5..2.5]): \; v = 0$

Constraint Newton Method

Projection Function Enclosure with the Interval Projection

The strategy used in the constraint Newton method is to search for the leftmost and the rightmost elements of the original variable domain satisfying the interval projection condition

Projection Function Enclosure based on the Interval Projection. Let $P=(X,D,C)$ be a CCSP. Let $c=(s,\rho)\in C$ be an n-ary constraint, B an n-ary F-box and x_i an element of s. Let a and b be respectively the leftmost and the rightmost elements of $B[x_i]$ satisfying the interval projection condition. The following property necessarily holds:

$$\pi_{x_i}^{\rho}(B) \subseteq [a..b]$$ ❑

What is needed is a function, denoted *narrowBounds*, with the following property:

$$\pi_{x_i}^{\rho}(B) \subseteq [a..b] \subseteq narrowBounds(B[x_i])$$

Constraint Newton Method

Projection Function Enclosure with the Interval Projection

To obtain a new bound, the projection condition is firstly verified in the extreme of the original domain and only in case of failure the leftmost (rightmost) zero of the interval projection is searched

function *narrowBounds*(an F-interval $[a..b]$)

if $a = b$ **then if** *intervalProjCond*($[a]$) **then return** $[a]$ **else return** $\varnothing$; **end if**; **end if**;

if not *intervalProjCond*($[a..a^+]$) **then** $a \leftarrow$ *searchLeft*($[a^+..b]$);

if $a = \varnothing$ **then return** $\varnothing$;

if $a = b$ **then return** $[b]$;

if not *intervalProjCond*($[b^-..b]$) **then** $b \leftarrow$ *searchRight*($[a..b^-]$);

return $[a..b]$;

end function

In case of failure of an inequality condition, it assumes that the leftmost (rightmost) element satisfying the interval projection condition must be a zero of the interval projection

Constraint Newton Method

Projection Function Enclosure with the Interval Projection

The verification if the interval projection condition is satisfied in a canonical interval is straightforward

function *intervalProjCond*(a canonical F-interval I)

(1) $[a..b] \leftarrow \prod_{x_i}^{\rho B}(I)$;

(2) **case** $\diamond$ **of**

(3) "=": **return** $0 \in [a..b]$;

(4) "≤": **return** $a \leq 0$;

(5) "≥": **return** $b \geq 0$;

(6) **end case**;

end function

Constraint Newton Method

Projection Function Enclosure with the Interval Projection

The algorithm for searching for the leftmost zero of an interval projection uses a Newton Narrowing function (*NN*) associated with the interval projection for reducing the search space

function *searchLeft*(an F-interval I)

$Q \leftarrow \{I\}$;
while $Q \neq \emptyset$ **do**
  choose $I_1 \in Q$ with the smallest left bound ($\forall_{I \in Q}\ left(I_1) \leq left(I)$);
  $Q \leftarrow Q \setminus \{I_1\}$;
  if $0 \in \prod_{x_i}^{\rho A}(I_1)$ **then**
    $I_1 \leftarrow NN(I_1)$;
    if $I_1 \neq \emptyset$ **then**
      $I_0 \leftarrow cleft(I_1)$; $I_1 \leftarrow [right(I_0)..right(I_1)]$;
      if $0 \in \prod_{x_i}^{\rho B}(I_0)$ **then return** $left(I_0)$;
      else $Q \leftarrow Q \cup \{[left(I_1)..\lfloor center(I_1) \rfloor], [\lfloor center(I_1) \rfloor..right(I_1)]\}$; **end if**;
    end if;
  end if;
end while;
return $\emptyset$;

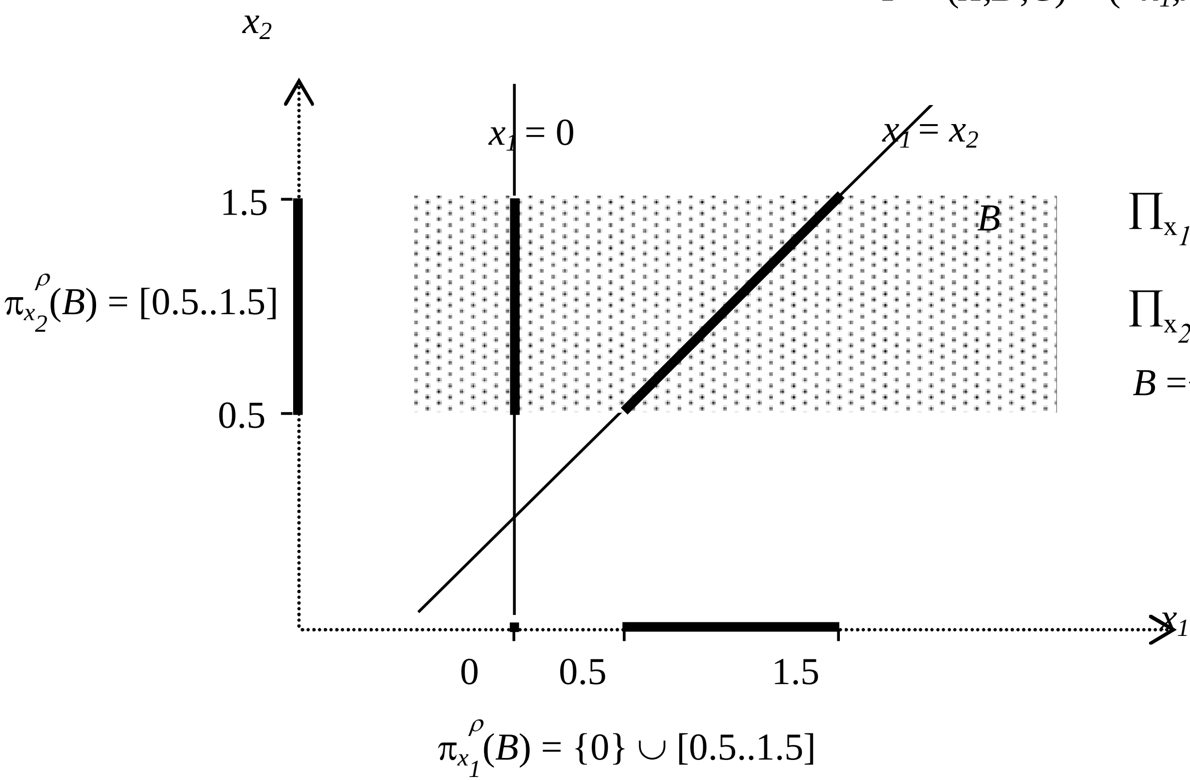
end function

Constraint Newton Method

Example

$P = (X,D,C) = (<x_1,x_2>, D_1 \times D_2, \{c\})$

$c \equiv x_1 \times (x_2 - x_1) = 0$



$\Pi_{x_1}^{\rho B} \equiv x_1 \times ([0.5..1.5] - x_1)$

$\Pi_{x_2}^{\rho B} \equiv [-0.5..2.5] \times (x_2 - [-0.5..2.5])$

$B = <[-0.5..2.5],[0.5..1.5]>$

Narrowing the domain of variable x_1: narrowBounds([-0.5,2.5])

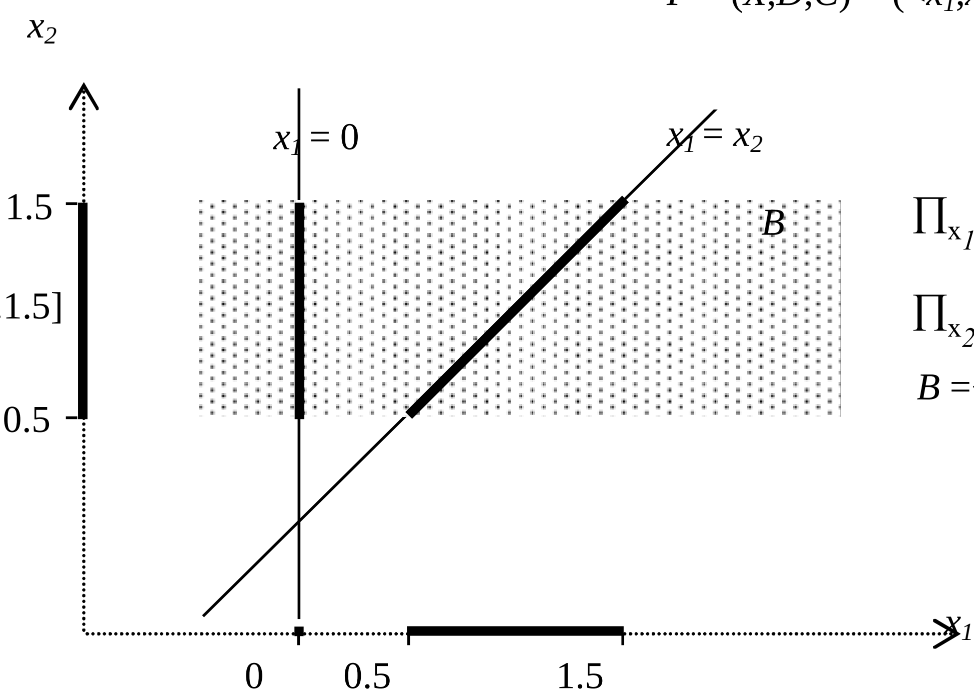
intervalProjCond([-0.5,-0.499]) → False $0 \notin \Pi_{x_1}^{\rho B}([-0.5,-0.499]) = [-1,-0.499]$

Constraint Newton Method

Example

$P = (X,D,C) = (\langle x_1,x_2\rangle, D_1\times D_2, \{c\})$

$c \equiv x_1\times(x_2-x_1) = 0$



$\pi^{\rho}_{x_2}(B) = [0.5..1.5]$

$\prod_{x_1}^{\rho B} \equiv x_1\times([0.5..1.5]-x_1)$

$\prod_{x_2}^{\rho B} \equiv [-0.5..2.5]\times(x_2-[-0.5..2.5])$

$B = \langle[-0.5..2.5],[0.5..1.5]\rangle$

searchLeft([-0.499..2.5])			
$Q=\{I_1,\ldots,I_n\}$	$0\in\prod_{x_i}^{\rho B}(I_1)$	$NN(I_1)$	$0\in\prod_{x_i}^{\rho B}(I_0)$
{[-0.499..2.5]}	0∈[-5..4.998]	[-0.499..2.5]	0∉[-0.998.. -0.497]
{[-0.498..1.001],[1.001..2.5]}	0∈[-0.995..2]	[-0.498..1.001]	0∉[-0.997.. -0.496]
{[-0.497..0.252],[0.252..1.001],[1.001..2.5]}	0∈[-0.992..0.504]	[0..0.001]	0∈[0..0.002]

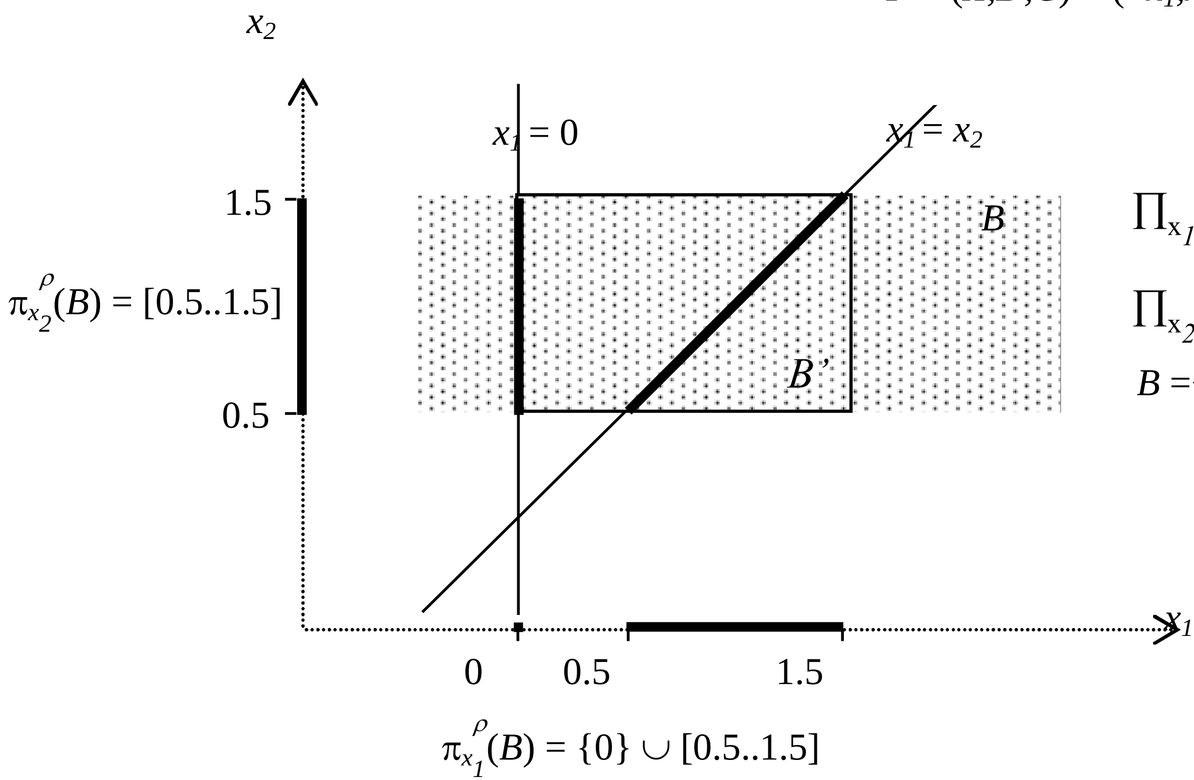
return 0

Constraint Newton Method

Example

$P = (X,D,C) = (\langle x_1,x_2\rangle,D_1\times D_2,\{c\})$

$c \equiv x_1\times(x_2\text{-}x_1) = 0$



$\prod_{x_1}^{\rho B} \equiv x_1\times([0.5..1.5]\text{-}x_1)$

$\prod_{x_2}^{\rho B} \equiv [-0.5..2.5]\times(x_2\text{-}[-0.5..2.5])$

$B = \langle[-0.5..2.5],[0.5..1.5]\rangle$

Proceeding similarly for the upper bound of x_1, the best narrowing is obtained:

$$B' = \langle[0.0,1.501],[0.5,1.0]\rangle$$

Complementary Approaches

A modification of the Newton method, is to use other interval extensions of the projection function associated with a constraint

A modification of the decomposition method, transforms the original set of constraints into an equivalent one where for each constraint (not necessarily primitive) the inverse interval expressions can be easily computed by interval arithmetic

Other modification is the introduction of a pre-processing phase preceding the definitions of the box-narrowing functions to obtain an equivalent CCSP (ex: introduction of redundant constraints)

Complementary Approaches

Other variation is the development of an algorithm capable of implementing a narrowing function for any constraint without decomposing, with the same results as the decomposition method

A complementary approach take advantage of the way that a complex constraint is expressed: An algorithm that does not require decomposing complex constraints, makes it possible to combine both basic methods, and choose either one or the other, according to the form of the expression of the interval projection

Finally, some approaches consider narrowing functions capable of narrowing several variable domains simultaneously (ex: based on the multivariate interval Newton method)