1. Interval Arithmetic

Consider the univariate polynomial function expressed in the standard form as: $f(x) = -4x^3 - x^2 + 2x$

1.1. Express this function in 2 equivalent forms:

a. Horner form $a + x(b + (c + dx)x)$

$x(2 + (-1 - 4x)x)$

b. Factored form (x-r1)(x-r2)(x-r3)

$x(x-(-1+\sqrt{33})/8)(x-(-1-\sqrt{33})/8)$

1.2. Compute the natural interval evaluation for each of the 3 forms for I=[0,1].

-4[0,1]^3-[0,1]^2+2[0,1]=
-4[0,1]-[0,1]+[0,2]=
[-4,0]-[0,1]+[0,2]=
[-5,0]+[0,2]=
[-5,2]

[0,1](2+(-1-4[0,1])[0,1])=
[0,1](2+(-1+[-4,0])[0,1])=
[0,1](2+[-5,-1][0,1])=
[0,1](2+[-5,0])=
[0,1][-3,2]=
[-3,2]

$[0,1]([0,1]-(-1+\sqrt{33})/8)([0,1]-(-1-\sqrt{33})/8)=$
[0,1]([0,1]-0.59307)([0,1]+0.84307)=
[0,1][-0.59307,0.40693][0.84307,1.84307]=
[-0.59307,0.40693][0.84307,1.84307]=
[-1.09307,0.75]

1.3. For Horner form and the Factored form choose an interval for which the natural interval evaluation computes the exact bounds of the function.

For the Horner form with the initial interval [-3,-2] the final lower bound (24) is always computed with the initial upper bound (-2) and the final upper bound (93) is always computed with the initial lower bound (-3):

[-3,-2](2+(-1-4[-3,-2])[-3,-2])=
[-3,-2](2+(-1+[8,12])[-3,-2])=
[-3,-2](2+[7,11][-3,-2])=
[-3,-2](2+[-33,-14])=
[-3,-2][-31,-12]=
[24,93]

For the Factored form with the initial interval [1,2] the final lower bound (0.75) is always computed with the initial lower bound (1) and the final upper bound (8) is always computed with the initial upper bound (2):

$[1,2]([1,2]-(-1+\sqrt{33})/8)([1,2]-(-1-\sqrt{33})/8)=$
[1,2]([1,2]-0.59307)([1,2]+0.84307)=
[1,2][0.40693,1.40693][1.84307,2.84307]=
[0.40693,2.81386][1.84307,2.84307]=
[0.75,8]

1.4. Define an algorithm that computes the exact bounds of the function for any I=[a,b].

First we will compute the values of x where the function where the function has a local maximum or a local minimum:

$f'(x) = -12x^2 - 2x + 2$

$-12x^2 - 2x + 2 = 0$

x=(2+10)/(-24) ∨ x=(2-10)/(-24)

x= -1/2 ∨ x=1/3

The algorithm in psudocode is:

```
lower = min(f(a),f(b))
if a ≤ -1/2 ≤ b then lower = min(lower,f(-1/2))
upper = max(f(a),f(b))
if a ≤ 1/3 ≤ b then upper = max(lower,f(1/3))
return [lower,upper]
```

2. Interval Newton

Consider the polynomial of the previous question: $f(x) = -4x^3 - x^2 + 2x$

2.1. Define the interval Newton function for the polynomial.

Let c([a,b])=(a+b)/2
N(I) = c(I)-(-4c(I)^3-c(I)^2+2c(I))/(-12I^2-2I+2)

2.2. Use the interval Newton method to prove that the polynomial has no roots in [0.75,1].

c([0.75,1])=0.875
N([0.75,1]) =
0.875-(-4(0.875)^3-(0.875)^2+2(0.875))/(-12[0.75,1]^2-2[0.75,1]+2)=
N([0.75,1]) = 0.875-(-2.6797-0.7656+1.75)/(-12[0.5625,1]-[1.5,2]+2)=
N([0.75,1]) = 0.875-(-1.6953)/([-12,-6.75]+[0,0.5])=
N([0.75,1]) = 0.875-(-1.6953)/([-12,-6.25])=
N([0.75,1]) = 0.875-(1.6953)/([6.25,12])=
N([0.75,1]) = 0.875-[0.1413,0.2712]=
N([0.75,1]) = [0.6038,0.7337]
[0.75,1] ⋂ N([0.75,1]) = [0.75,1] ⋂ [0.6038,0.7337] = ∅
therefore there are no roots in [0.75,1].

2.3. Can you prove with only 1 iteration of the interval Newton method that the polynomial has a root in [0.5,0.75]? Justify.

c([0.5,0.75])=0.625
N([0.5,0.75]) =
0.625-(-4(0.625)^3-(0.625)^2+2(0.625))/(-12[0.5,0.75]^2-2[0.5,0.75]+2)=
N([0.5,0.75]) = 0.625-(-0.9767-0.3906+1.25)/(-12[0.25,0.5625]-[1,1.5]+2)=
N([0.5,0.75]) = 0.625-(-0.1173)/([-6.75,-3]+[0.5,1])=
N([0.5,0.75]) = 0.625-(-0.1173)/([-6.25,-2])=
N([0.5,0.75]) = 0.625-(0.1173)/([2,6.25])=
N([0.5,0.75]) = 0.625-[0.0188,0.0586]=
N([0.5,0.75]) = [0.5664,0.6062] ⊂ [0.5,0.75]
therefore there is at least one root in [0.5,0.75].

3. Constraint Propagation

Consider the following Continuous Constraint Satisfaction Problem:

variables: x ∈ [0,1]
y ∈ [0,1]
constraints: 21x-18xy=5
15x+12y=14

3.1. Define a set of narrowing functions able to enforce hull-consistency on the constraints of the CCSP (for any box B ⊆ [0,1]^2).

Associated with constraint 21x-18xy=5

x = 5/(21-18y)

21-18y=5/x ⇔ 21-5/x=18y ⇔ y = 21/18-5/(18x)

Narrowing functions:

X = X ⋂ 5/(21-18Y)

Y = Y ⋂ 7/6-5/(18X)

Associated with constraint 15x+12y=14

x = (14-12y)/15

y = (14-15x)/12

Narrowing functions:

X = X ⋂ (14-12Y)/15

Y = Y ⋂ (14-15X)/12

3.2. Starting with the original domains box B ⊆ [0,1]^2, apply the above narrowing functions up to a fixed-point. What is the box obtained?

Using narrowing function X = X ⋂ 5/(21-18Y)

X = [0,1] ⋂ 5/(21-18[0,1])

X = [0,1] ⋂ 5/(21-[0,18])

X = [0,1] ⋂ 5/[3,21]

X = [0,1] ⋂ [5/21,5/3]

X = [5/21,1]

Using narrowing function Y = Y ⋂ 7/6-5/(18X)

Y = [0,1] ⋂ 7/6-5/(18[5/21,1])

Y = [0,1] ⋂ 7/6-5/[30/7,18]

Y = [0,1] ⋂ 7/6-[5/18,7/6]

Y = [0,1] ⋂ [0,8/9]

Y = [0,8/9]

Using narrowing function X = X ⋂ (14-12Y)/15

X = [5/21,1] ⋂ (14-12[0,8/9])/15

X = [5/21,1] ⋂ (14-[0,32/3])/15

X = [5/21,1] ⋂ [10/3,14]/15

X = [5/21,1] ⋂ [2/9,14/15]

X = [5/21,14/15]

Using narrowing function Y = Y ⋂ (14-15X)/12

Y = [0,8/9] ⋂ (14-15[5/21,14/15])/12

Y = [0,8/9] ⋂ (14-[25/7,14])/12

Y = [0,8/9] ⋂ [0,73/7]/12

Y = [0,8/9] ⋂ [0,73/84]

Y = [0,73/84]

Using narrowing function X = X ⋂ 5/(21-18Y)

X = [5/21,14/15] ⋂ 5/(21-18[0,73/84])

X = [5/21,14/15] ⋂ 5/(21-[0,219/14])

X = [5/21,14/15] ⋂ 5/[75/14,21]

X = [5/21,14/15] ⋂ [5/21,14/15]

X = [5/21,14/15]

Using narrowing function Y = Y ⋂ 7/6-5/(18X)

Y = [0,73/84] ⋂ 7/6-5/(18[5/21,14/15])

Y = [0,73/84] ⋂ 7/6-5/[30/7,84/5]

Y = [0,73/84] ⋂ 7/6-[25/84,7/6]

Y = [0,73/84] ⋂ [0,73/84]

Y = [0,73/84]

```
Using narrowing function X = X ⋂ (14-12Y)/15
   X = [5/21,14/15] ⋂ (14-12[0,73/84])/15
   X = [5/21,14/15] ⋂ (14-[0,146/14])/15
   X = [5/21,14/15] ⋂ [25/7,14]/15
   X = [5/21,14/15] ⋂ [5/21,14/15]
   X = [5/21,14/15]

Using narrowing function Y = Y ⋂ (14-15X)/12
   Y = [0,73/84] ⋂ (14-15[5/21,14/15])/12
   Y = [0,73/84] ⋂ (14-[25/7,14])/12
   Y = [0,73/84] ⋂ [0,73/7]/12
   Y = [0,73/84] ⋂ [0,73/84]
   Y = [0,73/84]

Fixed Point!
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3.3. Show the results that would be obtained
  during the execution of a branch-and-prune algorithm
     with the pruning step based on the above narrowing
  functions and a branching step that splits
     the largest variable domain in its midpoint. Start
  with the original domains box B ⊆ [0,1]^2
     and stop when the width of any variable domain is strictly smaller than 0.5.

     Starting with the initial domains:
        X = [0,1], Y = [0,1]
     And applying all narrowing functions
  up to a fixed point we obtain the box explained in 3.2.:
        X = [5/21,14/15], Y = [0,73/84]
     To proceed we have to branch spliting the largest domain.
        The domain width of x is 14/15 - 5/21 ≃ 0.6952
        The domain width of y is 73/84 - 0 ≃ 0.8690
     Therefore the next variable to split is y.
     So, the current box is split into boxes
        Left:  X = [5/21,14/15] ≃ [0.2381,0.9333], Y = [0,73/168] ≃ [0,0.4345]
        Right: X = [5/21,14/15] ≃
  [0.2381,0.9333], Y = [73/168,73/84] ≃ [0.4345,0.8690]

     Pruning the left box X = [5/21,14/15], Y = [0,73/168]:

     Using narrowing function X = X ⋂ 5/(21-18Y)
        X = [5/21,14/15] ⋂ 5/(21-18[0,73/168])
        X = [5/21,14/15] ⋂ 5/(21-[0,219/28])
        X = [5/21,14/15] ⋂ 5/[369/28,21]
        X = [5/21,14/15] ⋂ [5/21,140/369]
        X = [5/21,140/369] ≃ [0.2381,0.3794]

     Using narrowing function Y = Y ⋂ 7/6-5/(18X)
        Y = [0,73/168] ⋂ 7/6-5/(18[5/21,140/369])
        Y = [0,73/168] ⋂ 7/6-5/[30/7,280/41]
        Y = [0,73/168] ⋂ 7/6-[41/56,7/6]
        Y = [0,73/168] ⋂ [0,73/168]
        Y = [0,73/168] ≃ [0,0.4345]

     Using narrowing function X = X ⋂ (14-12Y)/15
        X = [5/21,140/369] ⋂ (14-12[0,73/168])/15
        X = [5/21,140/369] ⋂ (14-[0,73/14])/15
        X = [5/21,140/369] ⋂ [123/14,14]/15
        X = [5/21,140/369] ⋂ [41/70,14/15]
        X = ∅  (so there is no solution in this branch!)

     Pruning the right box X = [5/21,14/15], Y = [73/168,73/84]:
```

```
Using narrowing function X = X ⋂ 5/(21-18Y)
   X = [5/21,14/15] ⋂ 5/(21-18[73/168,73/84])
   X = [5/21,14/15] ⋂ 5/(21-[219/28,219/14])
   X = [5/21,14/15] ⋂ 5/[75/14,369/28]
   X = [5/21,14/15] ⋂ [140/369,14/15]
   X = [140/369,14/15] ≃ [0.3794,0.9333]
Using narrowing function Y = Y ⋂ 7/6-5/(18X)
   Y = [73/168,73/84] ⋂ 7/6-5/(18[140/369,14/15])
   Y = [73/168,73/84] ⋂ 7/6-5/[280/41,84/5]
   Y = [73/168,73/84] ⋂ 7/6-[25/84,41/56]
   Y = [73/168,73/84] ⋂ [73/168,73/84]
   Y = [73/168,73/84] ≃ [0.4345,0.8690]
Using narrowing function X = X ⋂ (14-12Y)/15
   X = [140/369,14/15] ⋂ (14-12[73/168,73/84])/15
   X = [140/369,14/15] ⋂ (14-[73/14,73/7])/15
   X = [140/369,14/15] ⋂ [25/7,123/14]/15
   X = [140/369,14/15] ⋂ [5/21,41/70]
   X = [140/369,41/70] ≃ [0.3794,0.5857]
Using narrowing function Y = Y ⋂ (14-15X)/12
   Y = [73/168,73/84] ⋂ (14-15[140/369,41/70])/12
   Y = [73/168,73/84] ⋂ (14-[700/123,123/14])/12
   Y = [73/168,73/84] ⋂ [73/14,1022/123]/12
   Y = [73/168,73/84] ⋂ [73/168,511/738]
   Y = [73/168,73/84] ≃ [0.4345,0.8690]
Using narrowing function X = X ⋂ 5/(21-18Y)
   X = [140/369,41/70] ⋂ 5/(21-18[73/168,73/84])
   X = [140/369,41/70] ⋂ [140/369,14/15] (see previous)
   X = [140/369,41/70] ≃ [0.3794,0.9333]
Using narrowing function Y = Y ⋂ 7/6-5/(18X)
   Y = [73/168,73/84] ⋂ 7/6-5/(18[140/369,41/70])
   Y = [73/168,73/84] ⋂ 7/6-5/[280/41,369/35]
   Y = [73/168,73/84] ⋂ 7/6-[175/369,41/56]
   Y = [73/168,73/84] ⋂ [73/168,511/738]
   Y = [73/168,73/84] ≃ [0.4345,0.8690]
Using narrowing function X = X ⋂ (14-12Y)/15
   X = [140/369,41/70] ⋂ (14-12[73/168,73/84])/15
   X = [140/369,41/70] ⋂ [5/21,41/70]  (see previous)
   X = [140/369,41/70] ≃ [0.3794,0.5857]
Fixed Point!
Therefore, the final result is the single box (both domains have width <0.5):
   X = [140/369,41/70] ≃ [0.3794,0.5857], Y = [73/168,73/84] ≃ [0.4345,0.8690]
```