Interval Constraints Overview

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Nov 2015

Lecture 1: Interval Constraints Overview

Continuous Constraint Satisfaction Problems

Continuous Constraint Reasoning

- Representation of Continuous Domains
- Pruning and Branching

Solving Continuous CSPs

- Constraint Propagation
- Consistency Criteria

Practical Examples

Course Structure

Constraint Reasoning

Continuous CSP (CCSP):

Constraint Satisfaction Problem (CSP):

set of variables

set of domains → Intervals of reals [a,b]

set of constraints → Numeric (=,≤,≥)

Solution: → Many

assignment of values which satisfies all the constraints

GOAL Find Solutions;
Find an enclosure of the solution space

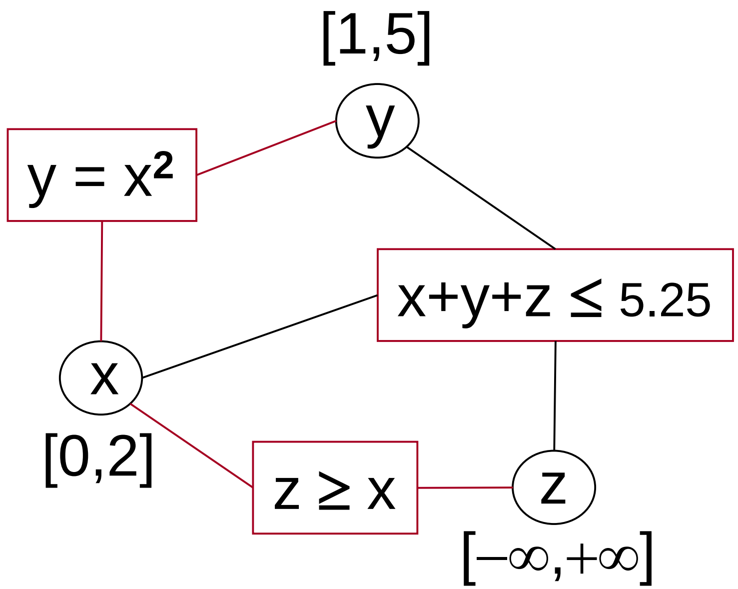
Constraint Reasoning

Continuous Constraint Satisfaction Problem (CCSP):

Interval Domains

Numerical Constraints

Many Solutions



Solution:

x=1, y=1, z=1

...

x=1, y=1, z=3.25

...

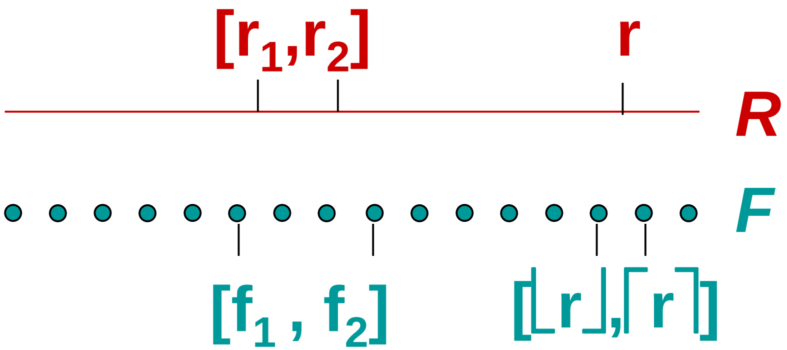
assignment of values which satisfies all the constraints

GOAL Find solutions;

Find an enclosure of the solution space

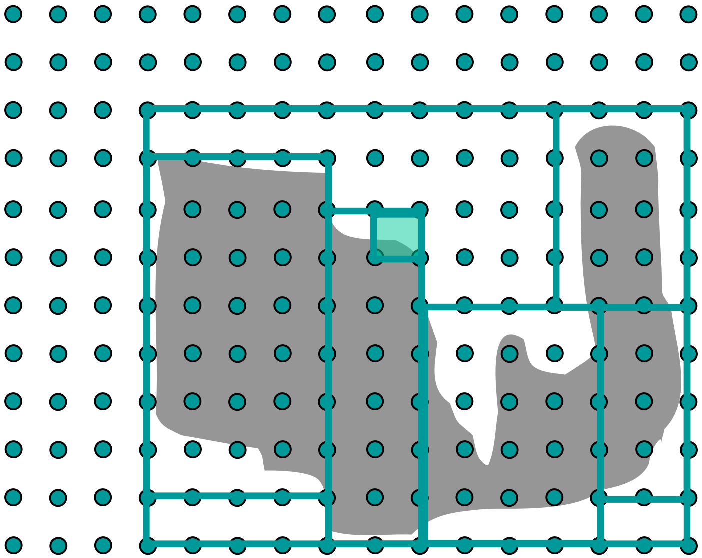
Representation of Continuous Domains

***F*-interval**



***F*-box**

Canonical solution

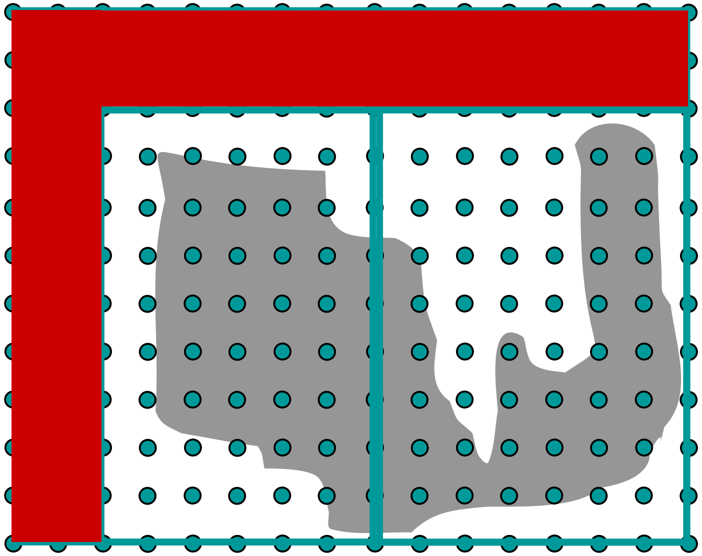


Solving CCSPs:

Branch and Prune algorithms

constraint propagation

box split



Safe Narrowing Functions

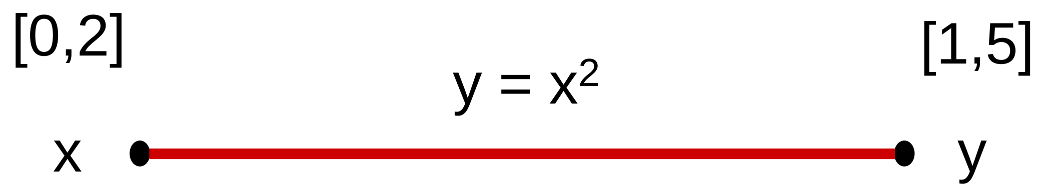
Strategy for
- isolate canonical solutions
- provide an enclosure of the solution space

depends on a consistency requirement

Constraint Reasoning (vs Simulation)

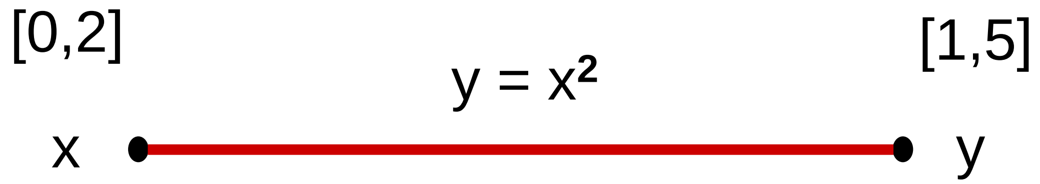
Represents uncertainty as intervals of possible values

Uses safe methods for narrowing the intervals accordingly to the constraints of the model



		x	y	
Simulation:		0	0	no
	x≤1?	1	1	
		2	4	y≥4?
Constraint Reasoning:		[1,2]	[1,4]	

How to narrow the domains?



Safe methods are based on Interval Analysis techniques

$$\forall_{x\in[a,b]}\; x^2 \in [a,b]^2 = \begin{cases} [0, \lceil \max(a^2,b^2) \rceil] & \text{if } a \le 0 \le b \\ [\lfloor \min(a^2,b^2) \rfloor, \lceil \max(a^2,b^2) \rceil] & \text{otherwise} \end{cases}$$

If $x \in [0,2]$

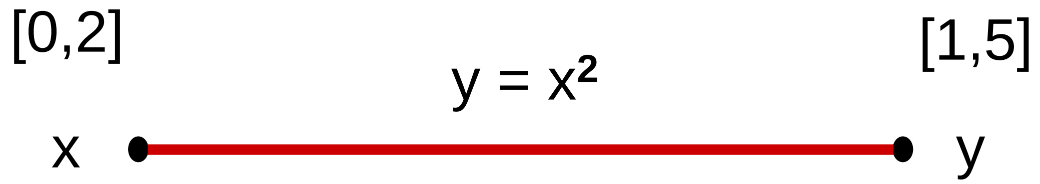
Then $y \in [0,2]^2 = [0, \lceil \max(0^2,2^2) \rceil] = [0,4]$

$\therefore\; y \in [1,5] \wedge y \in [0,4]$

$\therefore\; y \in [1,5] \cap [0,4]$

$\therefore\; y \in [1,4]$

How to narrow the domains?



Safe methods are based on Interval Analysis techniques

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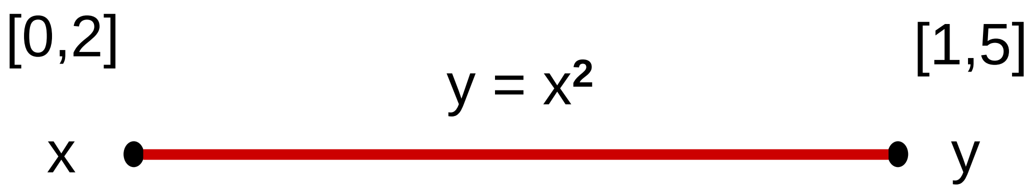
$$NF_{y=x^2}: Y' \leftarrow Y \cap X^2$$

How to narrow the domains?



Safe methods are based on Interval Analysis techniques

$y - x^2 = 0 \longrightarrow F(Y) = Y - [0,2]^2 \qquad F'(Y) = 1$

$$\forall_{y \in Y} \forall_{x \in [0,2]} \; y - x^2 = 0 \Rightarrow y \in N(Y) = c(Y) - \frac{F(c(Y))}{F'(Y)}$$

Interval Newton method

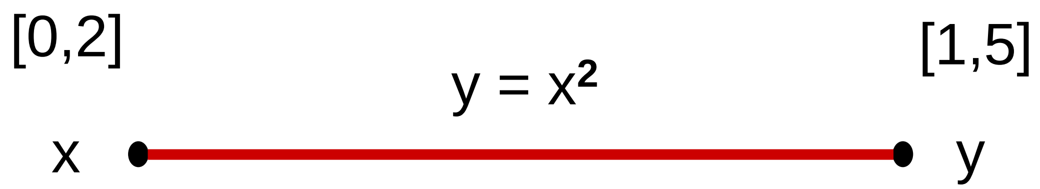
If $x \in [0,2]$ and $y \in [1,5]$

Then $y \in N([1,5]) = 3 - \dfrac{3 - [0,2]^2}{1} = [0,4]$

$\therefore \; y \in [1,5] \cap [0,4]$

$\therefore \; y \in [1,4]$

How to narrow the domains?



Safe methods are based on Interval Analysis techniques

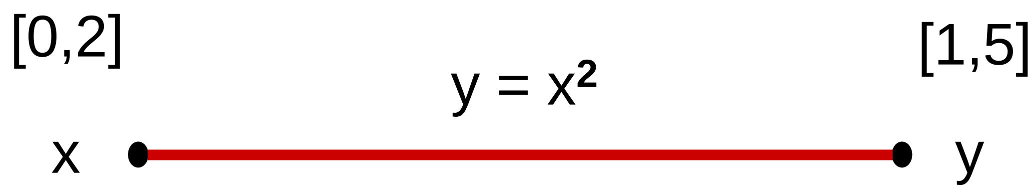
$y - x^2 = 0 \longrightarrow F(Y) = Y - [0,2]^2 \qquad F'(Y) = 1$

$$\forall_{y \in Y} \forall_{x \in [0,2]}\ y - x^2 = 0 \Rightarrow y \in N(Y) = c(Y) - \frac{F(c(Y))}{F'(Y)}$$

Interval Newton method

$$NF_{y=x^2}:\ Y' \leftarrow Y \cap \left(c(Y) - \frac{c(Y) - X^2}{1} \right)$$

How to narrow the domains?

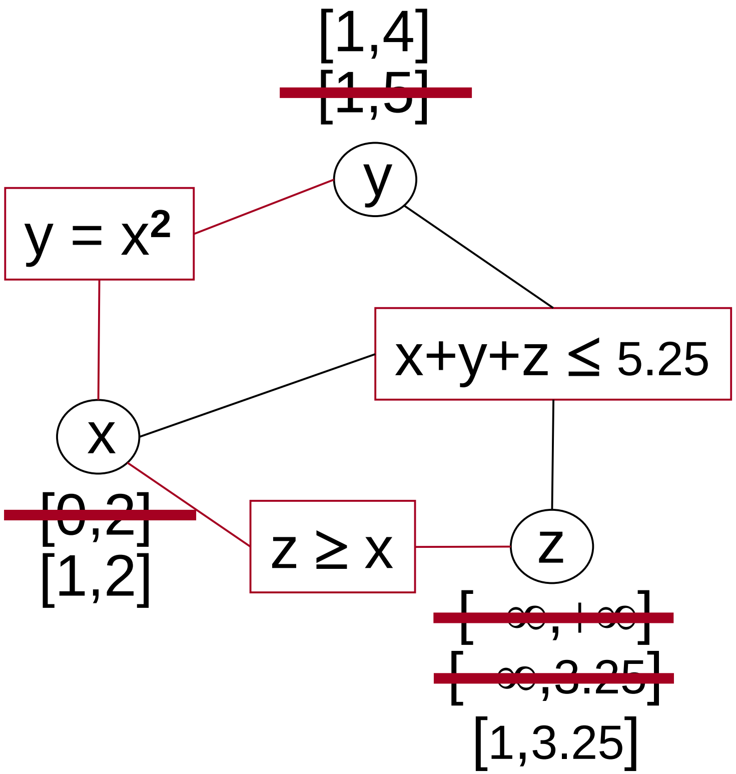


Safe methods are based on Interval Analysis techniques

	contractility	correctness
$NF_{y=x^2}: Y' \leftarrow Y \cap X^2$ $NF_{y=x^2}: Y' \leftarrow Y \cap \left(c(Y) - \dfrac{c(Y) - X^2}{1} \right)$	$Y' \subseteq Y$	$\forall_{y \in Y}\, y \notin Y' \Rightarrow \neg\exists_{x \in X}\, y = x^2$
$NF_{y=x^2}: X' \leftarrow (X \cap -Y^{1/2}) \uplus (X \cap +Y^{1/2})$ $NF_{y=x^2}: X' \leftarrow X \cap \left(c(X) - \dfrac{Y - c(X)^2}{-2X} \right)$	$X' \subseteq X$	$\forall_{x \in X}\, x \notin X' \Rightarrow \neg\exists_{y \in Y}\, y = x^2$

Solving a Continuous Constraint Satisfaction Problem

Constraint Propagation



$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

→ √ $NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{½}) \uplus (X \cap +Y^{½})$

$NF_{x+y+z \leq 5.25}$: $X' \leftarrow X \cap ([-\infty, 5.25] - Y - Z)$

$NF_{x+y+z \leq 5.25}$: $Y' \leftarrow Y \cap ([-\infty, 5.25] - X - Z)$

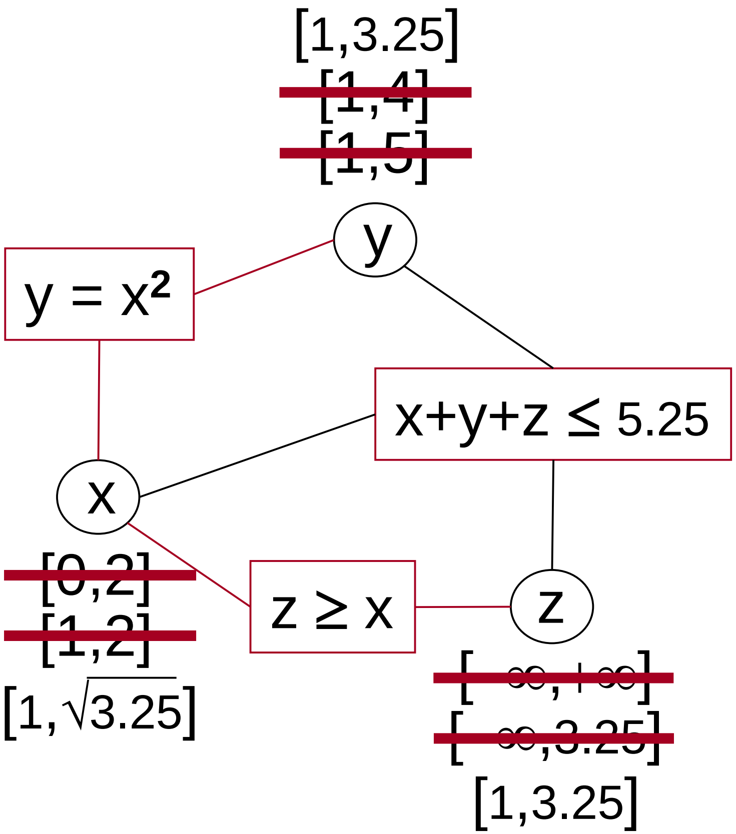
→ √ $NF_{x+y+z \leq 5.25}$: $Z' \leftarrow Z \cap ([-\infty, 5.25] - X - Y)$

→ √ $NF_{z \geq x}$: $X' \leftarrow X \cap (Z - [0, +\infty])$

→ √ $NF_{z \geq x}$: $Z' \leftarrow Z \cap (X + [0, +\infty])$

Solving a Continuous Constraint Satisfaction Problem

Constraint Propagation



√ $NF_{y=x^2}: Y' \leftarrow Y \cap X^2$

√ $NF_{y=x^2}: X' \leftarrow (X \cap -Y^{½}) \uplus (X \cap +Y^{½})$

√ $NF_{x+y+z \leq 5.25}: X' \leftarrow X \cap ([-\infty, 5.25] - Y - Z)$

√ $NF_{x+y+z \leq 5.25}: Y' \leftarrow Y \cap ([-\infty, 5.25] - X - Z)$

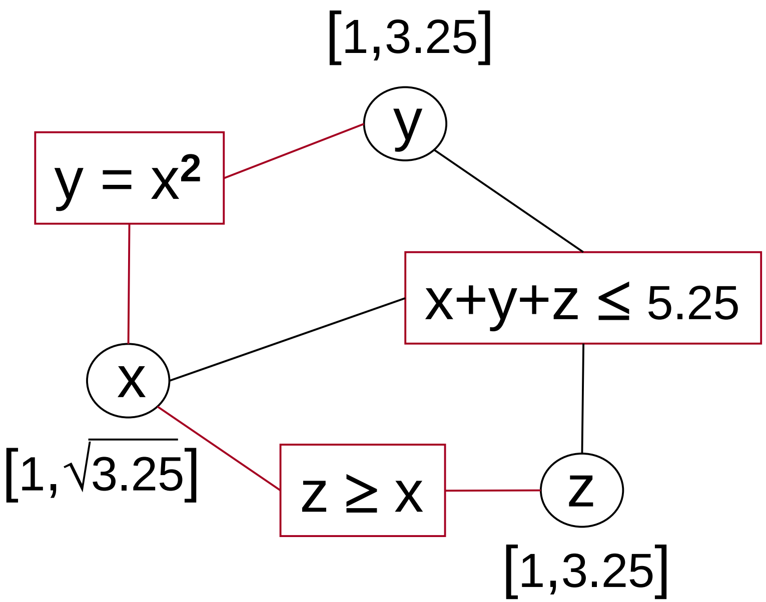
√ $NF_{x+y+z \leq 5.25}: Z' \leftarrow Z \cap ([-\infty, 5.25] - X - Y)$

√ $NF_{z \geq x}: X' \leftarrow X \cap (Z - [0, +\infty])$

√ $NF_{z \geq x}: Z' \leftarrow Z \cap (X + [0, +\infty])$

Solving a Continuous Constraint Satisfaction Problem

- Constraint Propagation + Branching
- Consistency Criterion



x	y	z	
1	1	1	√
1	1	3.25	√
$\sqrt{3.25}$			X
1.5	2.25		√

$y = x^2 \Rightarrow y = 3.25$

$x+y+z \leq 5.25 \Rightarrow z \leq 2-\sqrt{3.25}$ ($2-\sqrt{3.25} < \sqrt{3.25}$)

~~z ≥ x~~

Solving a Continuous Constraint Satisfaction Problem

{ Constraint Propagation + Branching
  Consistency Criterion

Local Consistency (2B-Consistency) ← Constraint Propagation

↓

Higher Order Consistencies (kB-Consistency) ← Constraint Propagation + Branching

3B-Consistency: if 1 bound is fixed then the problem is Local Consistent

x	y	z			x	y	z
$[1,\sqrt{3.25}]$	[1,3.25]	[1,3.25]	not 3B-Consistent	←	$[\sqrt{3.25}]$	[1,3.25]	[1,3.25]
[1,1.5]	[1,2.25]	[1,3.25]	3B-Consistent		not Local Consistent		

Example:

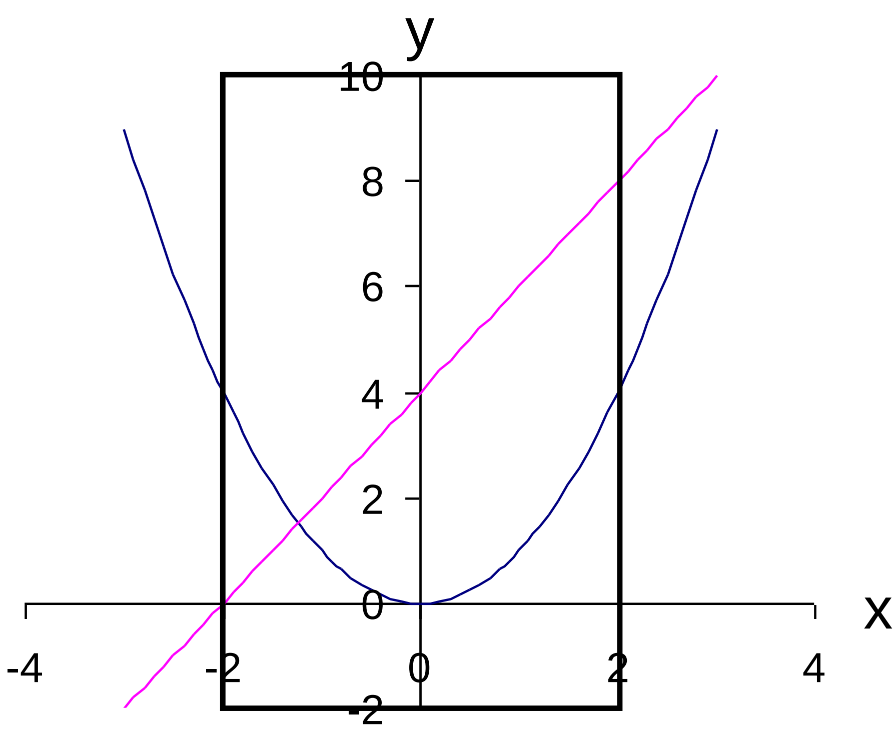
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

define set of narrowing functions:

$y = x^2$ → $NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$x = \pm y^{½}$ → $NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{½}) \uplus (X \cap +Y^{½})$

$y = 2x + [4,+\infty]$ → $NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$x = ½y - [2,+\infty]$ → $NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (½Y - [2,+\infty])$

Example:

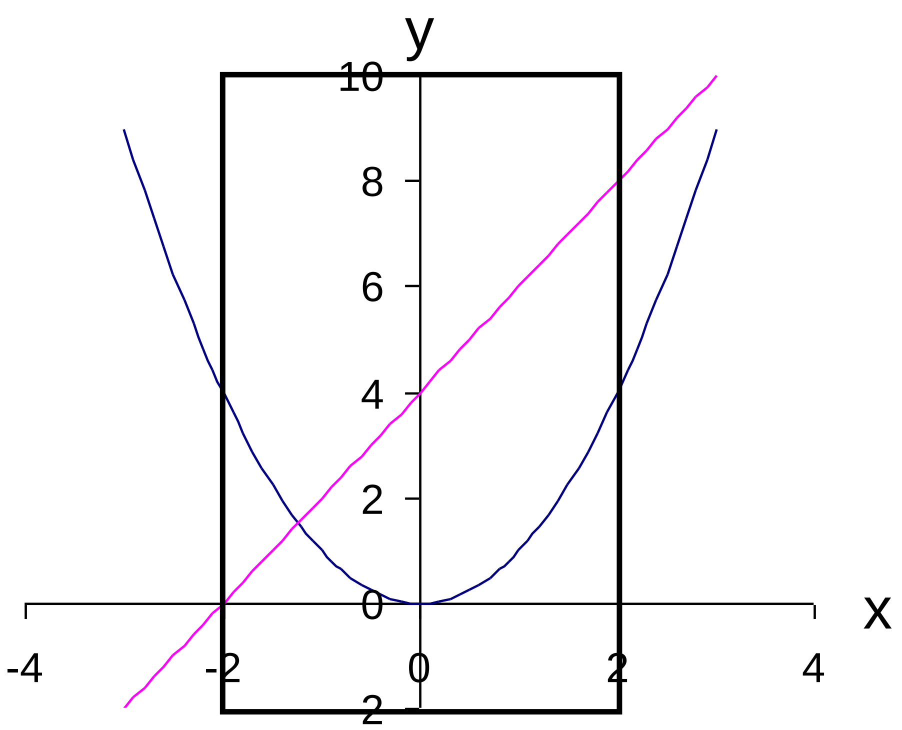
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: $[-2,2]\times[-2,10]$

$[-2,2]\times([-2,10]\cap[-2,2]^2)$ ← $NF_{y=x^2}: Y' \leftarrow Y \cap X^2$

$[-2,2]\times([-2,10]\cap[0,4])$ — $NF_{y=x^2}: X' \leftarrow (X\cap -Y^{½}) \uplus (X \cap +Y^{½})$

$[-2,2]\times[0,4]$ — $NF_{y\geq 2x+4}: Y' \leftarrow Y\cap(2X+[4,+\infty])$

$NF_{y\geq 2x+4}: X' \leftarrow X\cap(½Y-[2,+\infty])$

Example:

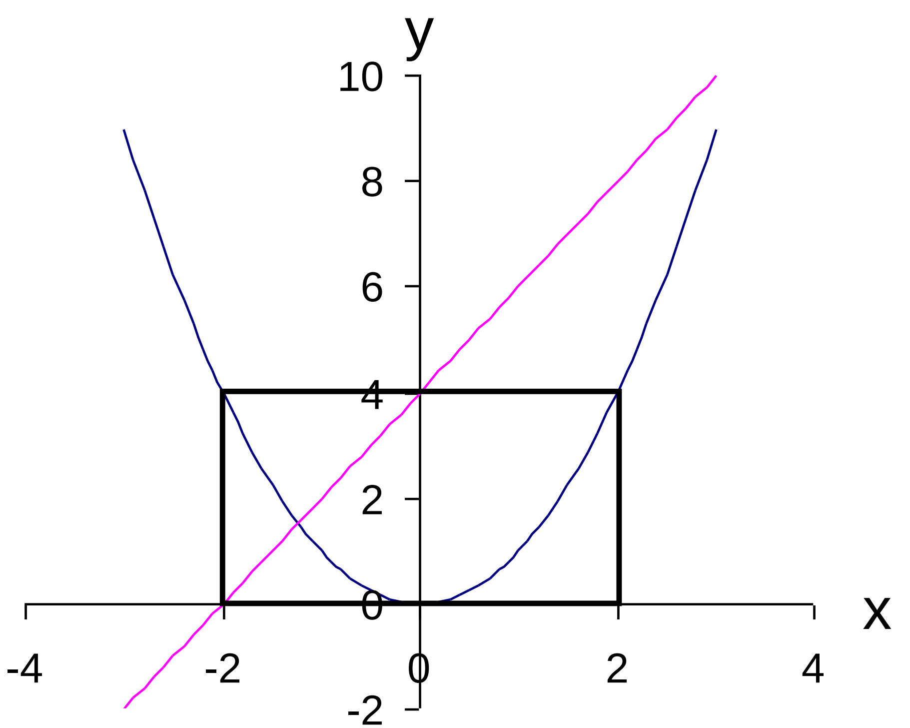
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: [-2,2]×[0,4]

$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{1/2}) \cup (X \cap +Y^{1/2})$

$NF_{y \geq 2x+4}$: $Y' \leftarrow Y \cap (2X + [4, +\infty])$

$NF_{y \geq 2x+4}$: $X' \leftarrow X \cap (\frac{1}{2}Y - [2, +\infty])$

$([-2,2] \cap -[0,4]^{1/2}) \dot{\cup} ([-2,2] \cap [0,4]^{1/2}) \times [0,4]$

$([-2,2] \cap [-2,0]) \dot{\cup} ([-2,2] \cap [0,2]) \times [0,4]$ ←

$[-2,2] \times [0,4]$

Example:

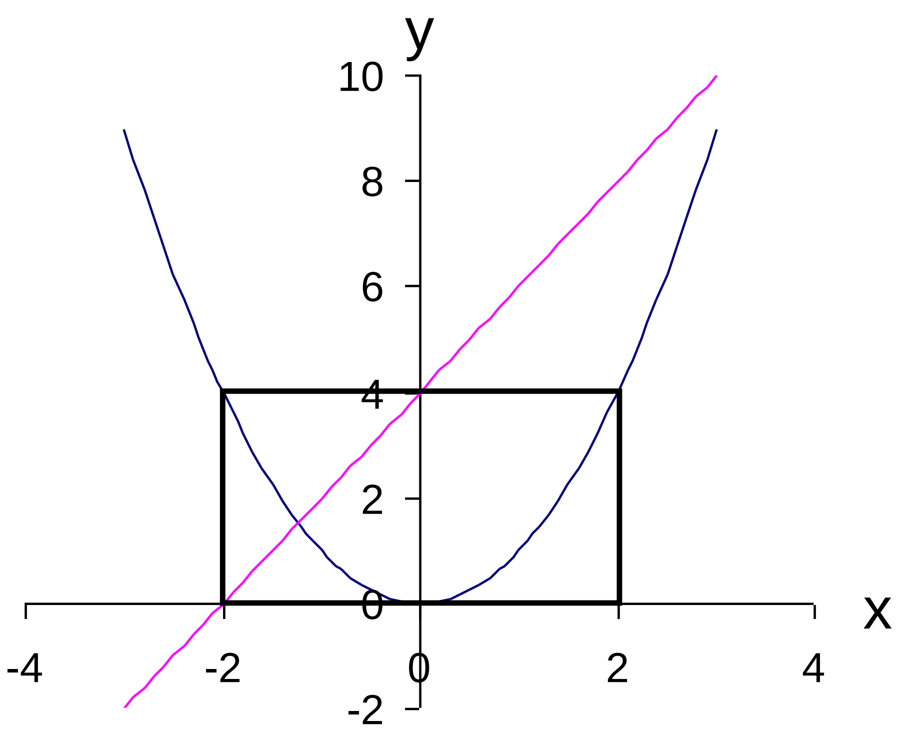
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: $[-2,2]\times[0,4]$

$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{1/2}) \cup (X \cap +Y^{1/2})$

$NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (1/2 Y - [2,+\infty])$

$[-2,2]\times([0,4]\cap(2[-2,2]+[4,+\infty]))$

$[-2,2]\times([0,4]\cap([-4,4]+[4,+\infty]))$

$[-2,2]\times([0,4]\cap[0,+\infty])$ ←

$[-2,2]\times[0,4]$

Example:

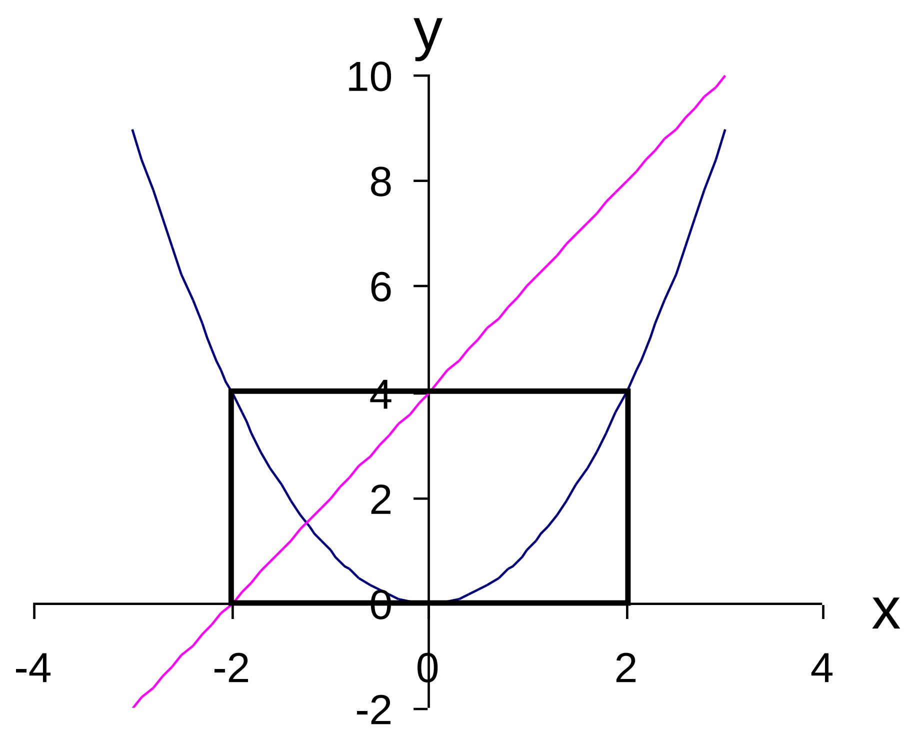
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: $[-2,2]\times[0,4]$

$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{½}) \cup (X \cap +Y^{½})$

$NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (½Y-[2,+\infty])$

$([-2,2]\cap(½[0,4]-[2,+\infty]))\times[0,4]$

$([-2,2]\cap([0,2]-[2,+\infty]))\times[0,4]$

$([-2,2]\cap[-\infty,0])\times[0,4]$

$[-2,0]\times[0,4]$

Example:

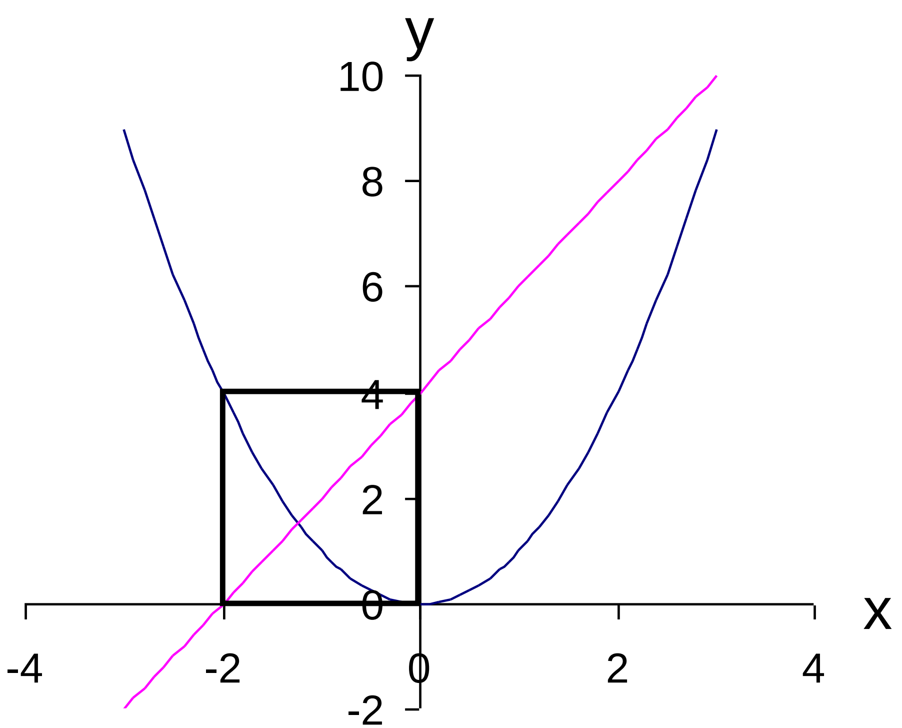
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

obtained the box: $[-2,0]\times[0,4]$ **(fixed point)**

$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{½}) \cup (X \cap +Y^{½})$

$NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (½Y - [2,+\infty])$

Example:

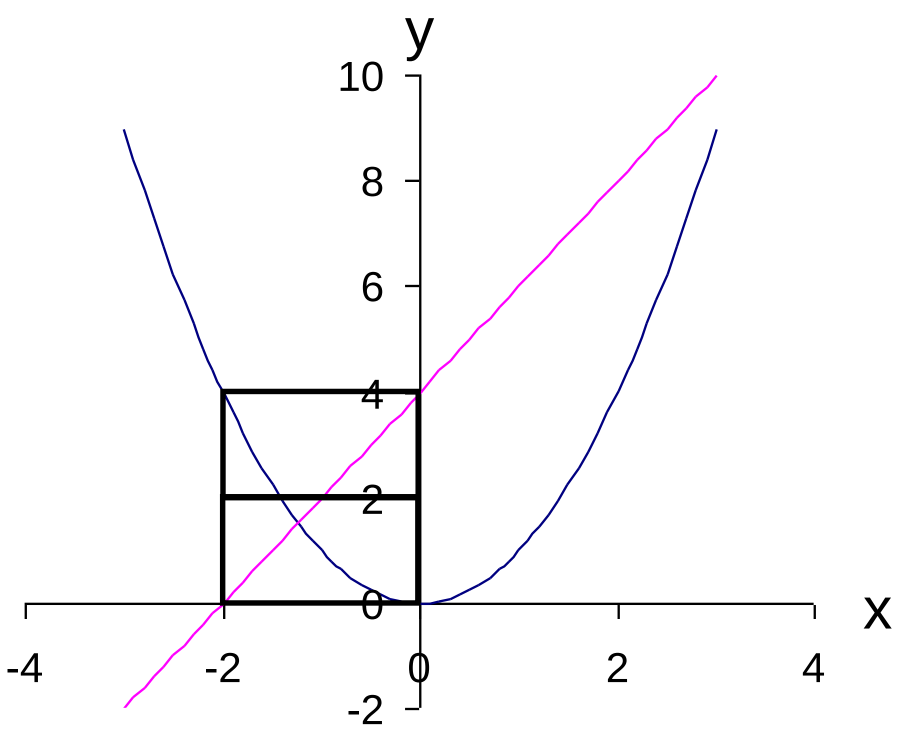
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Split box

$[-2,0]\times[0,2]$

$[-2,0]\times[2,4]$

Example:

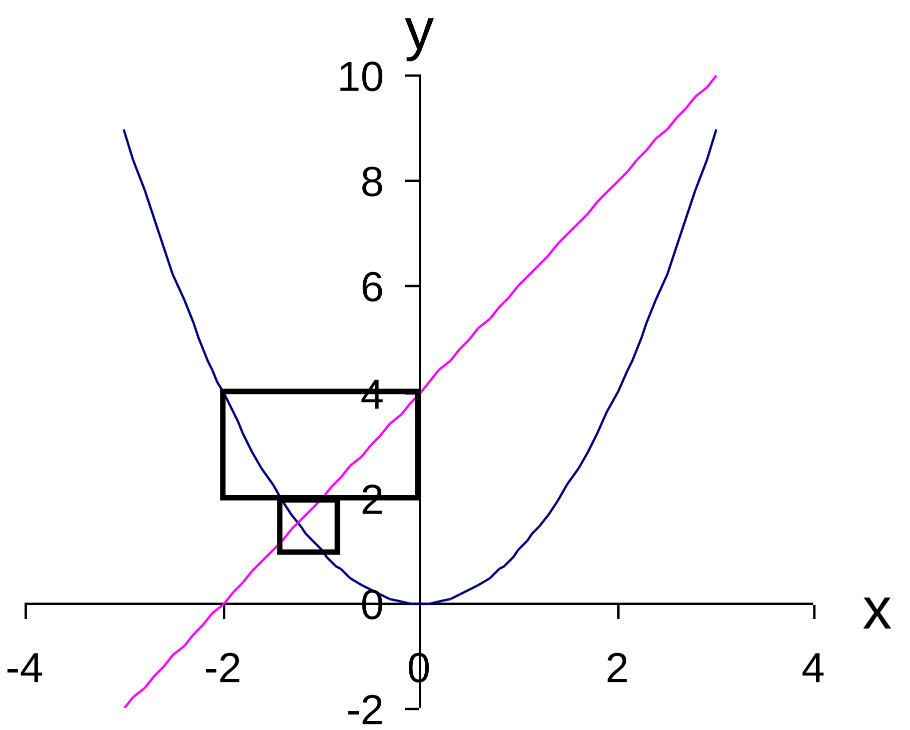
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Split box

[-2,0]×[0,2] —prune→ [-1.415,-1.082]×[1.171,2.000] (fixed point)

[-2,0]×[2,4]

Example:

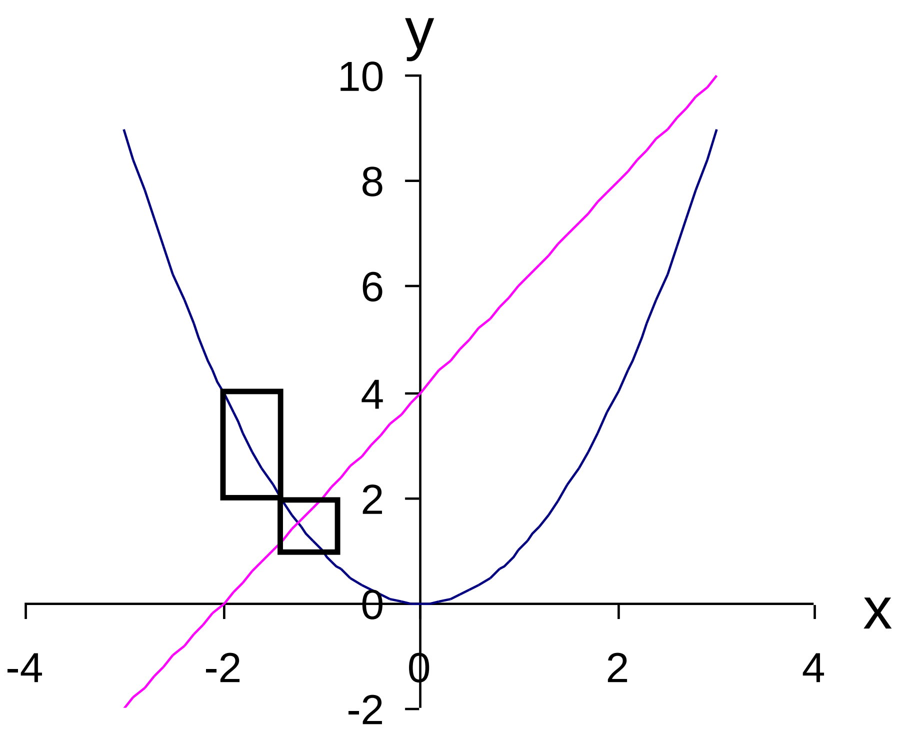
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Split box

[-2,0]×[0,2] —prune→ [-1.415,-1.082]×[1.171,2.000] (fixed point)

[-2,0]×[2,4] —prune→ [-2.000,-1.414]×[2.000,4.000] (fixed point)

Example:

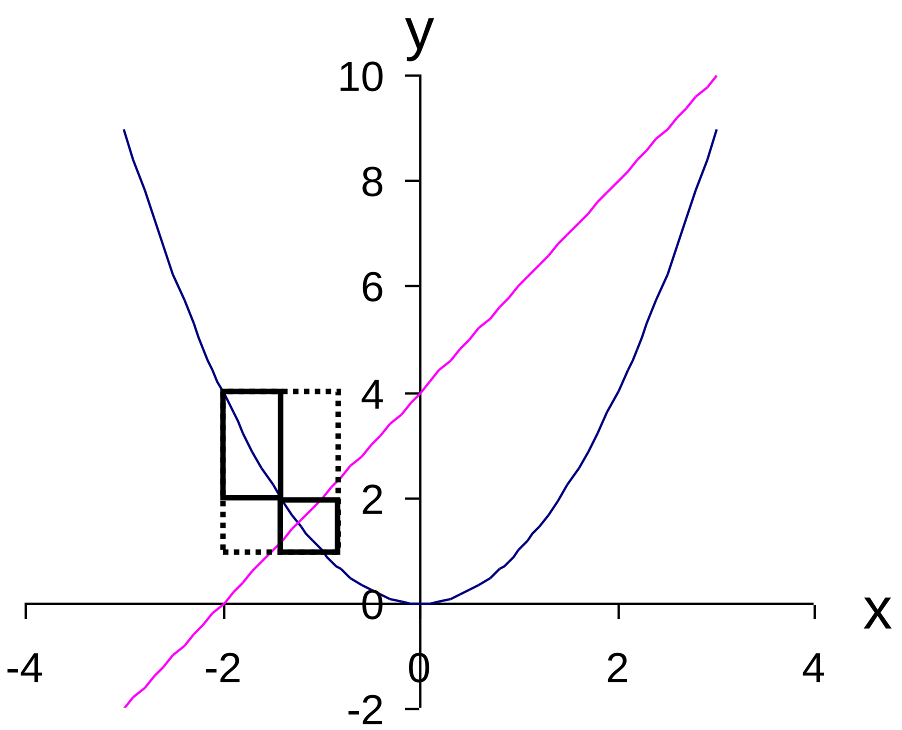
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



When to stop? → Consistency requirement

if we stop now:

$[-1.415,-1.082]\times[1.171,2] \uplus [-2,-1.414]\times[2,4] = [-2\ ,-1.082]\times[1.171,4]$

Example:

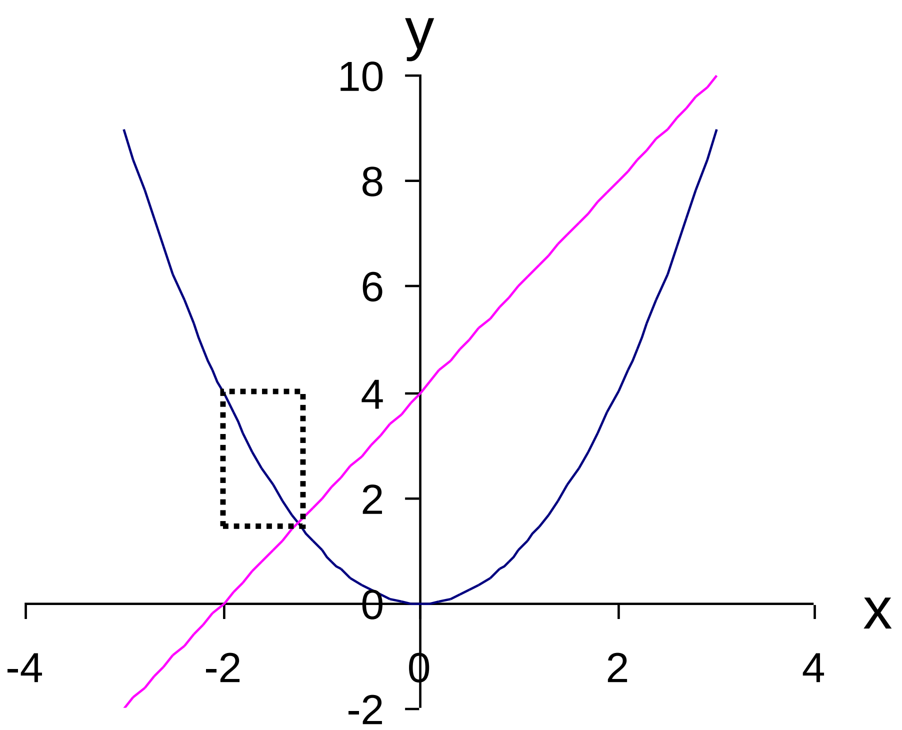
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

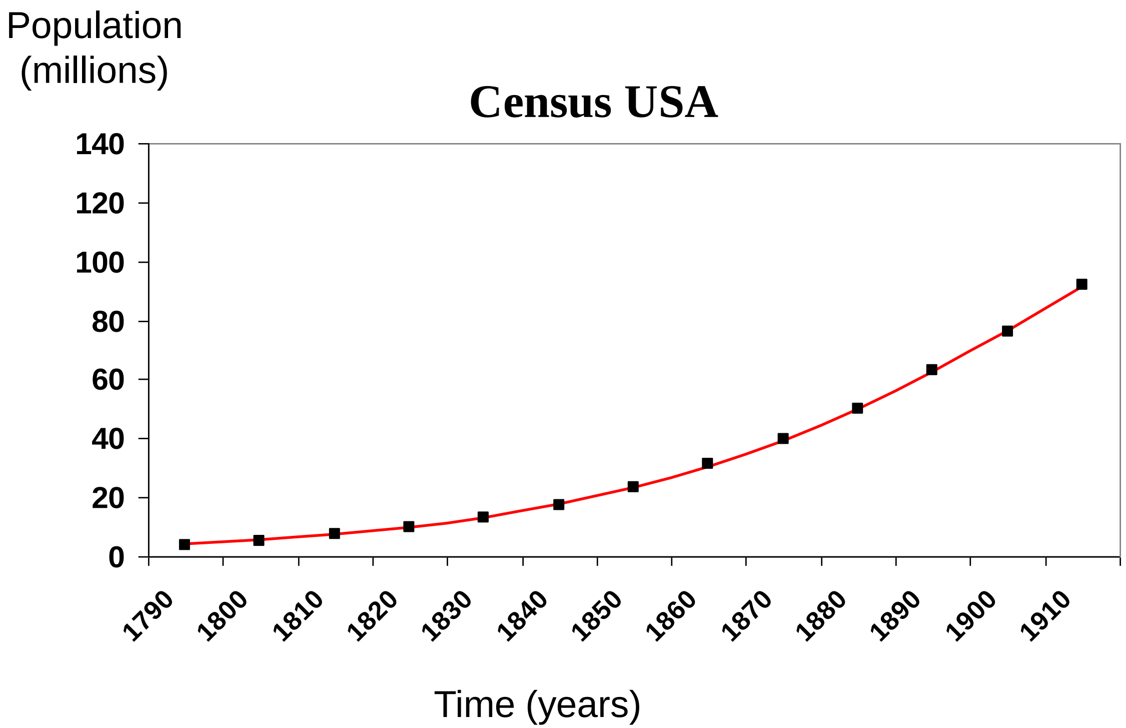
$y \geq 2x + 4$



When to stop? → Consistency requirement

smallest box containing all canonical solutions

A practical example:



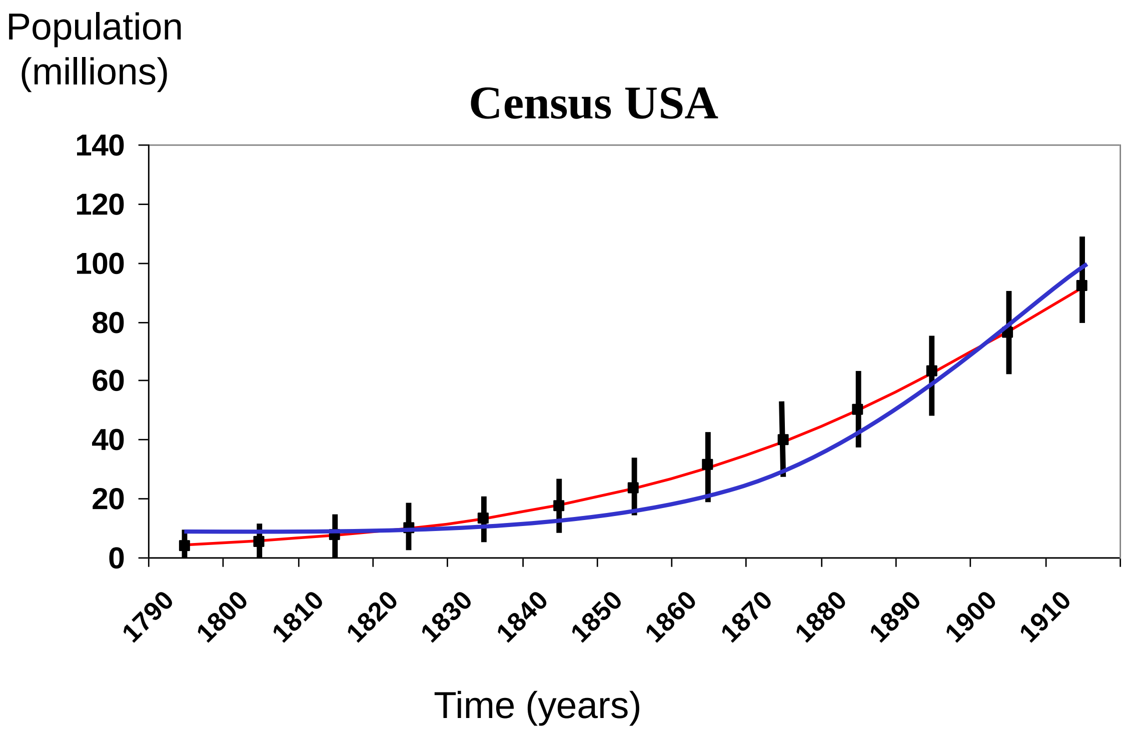
Logistic Model

$$x(t) = \frac{k x_0 e^{r(t-t_0)}}{x_0\left(e^{r(t-t_0)} - 1\right) + k}$$

result: $\begin{cases} x_0 = a \\ k = b \\ r = c \end{cases}$

Optimization Problem: $min \sum_i (x_i - v_i)^2$ with $x_i = \frac{k x_0 e^{r(t_i-t_0)}}{x_0\left(e^{r(t_i-t_0)} - 1\right) + k}$

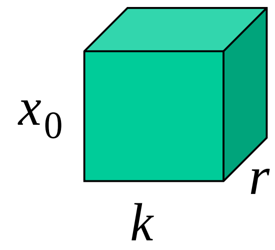
A practical example:



Logistic Model

$$x(t) = \frac{k x_0 e^{r(t-t_0)}}{x_0\left(e^{r(t-t_0)} - 1\right) + k}$$

result:



CCSP: $$\left\{ \langle x_0, k, r \rangle \mid \forall_{(t_i, v_i)}\, x_i = \frac{k x_0 e^{r(t_i - t_0)}}{x_0\left(e^{r(t_i - t_0)} - 1\right) + k} \quad \wedge \quad |x_i - v_i| \le \varepsilon_i \right\}$$

Course Structure: Constraints on Continuous Domains

Lecture 1: Interval Constraints Overview

Lecture 2: Intervals, Interval Arithmetic and Interval Functions

Lecture 3: Interval Newton Method

Lecture 4: Associating Narrowing Functions to Constraints

Lecture 5: Constraint Propagation and Consistency Enforcement

Lecture 6: Problem Solving

Bibliography

- Jorge Cruz. *Constraint Reasoning for Differential Models*
  Vol: 126 Frontiers in Artificial Intelligence and Applications, IOS Press 2005
- Ramon E. Moore. *Interval Analysis*
  Prentice-Hall 1966
- Eldon Hansen, G. William Walster. *Global Optimization Using Interval Analysis*
  Marcel Dekker 2003
- Jaulin, L., Kieffer, M., Didrit, O., Walter, E. *Applied Interval Analysis*
  Springer 2001

Important Links

- *Interval Computations*
  A primary entry point to items concerning interval computations.
- *COCONUT - COntinuous CONstraints Updating the Technology*
  Project to integrate techniques from mathematical programming, constraint programming, and interval analysis.

Papers

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- F. Benhamou and W. Older. Applying interval arithmetic to real, integer and boolean constraints. Journal of Logic Programming, pages 1–24, 1997.
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